Reachability in Two-Dimensional Vector Addition Systems with States is PSPACE-complete

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June 15, 2015

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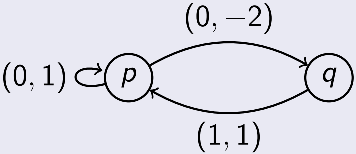
Vector addition system with states (VASS)

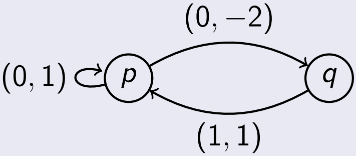
d-VASS:



Vector addition system with states (VASS)

d-VASS:

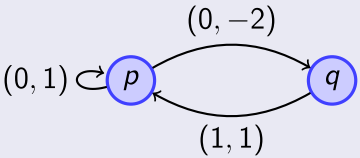
- $\boldsymbol{d \geq 1}$ ***(dimension)***



Vector addition system with states (VASS)

d-VASS:

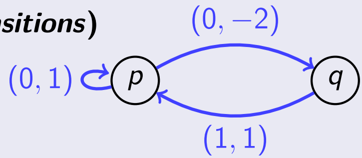
- $d \geq 1$ (*dimension*)
- **Q finite set (*states*)**



Vector addition system with states (VASS)

d-VASS:

- $d \geq 1$ (*dimension*)
- Q finite set (*states*)
- $\boldsymbol{T \subseteq Q \times \mathbb{Z}^d \times Q}$ **finite (*transitions*)**



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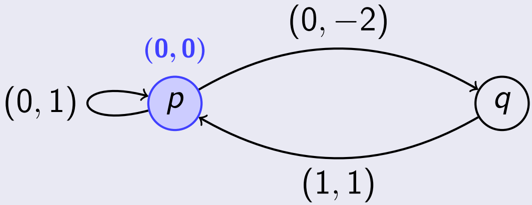

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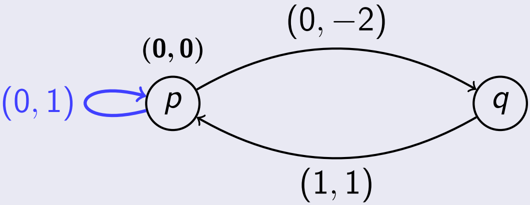

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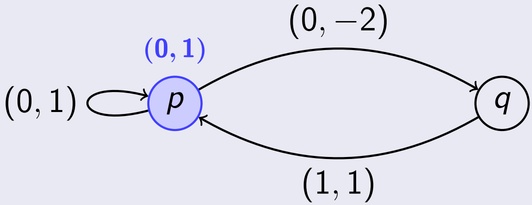

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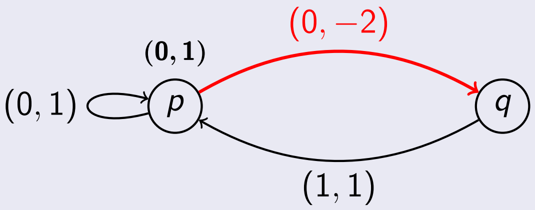

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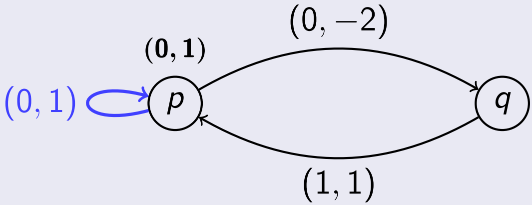

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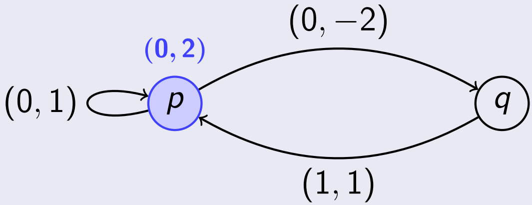

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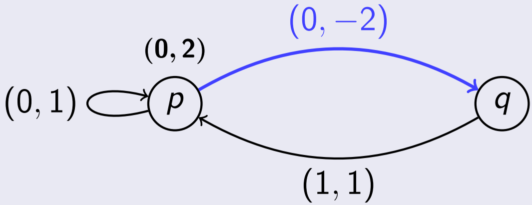

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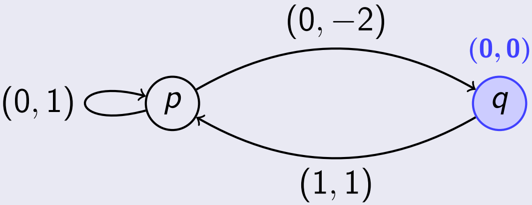

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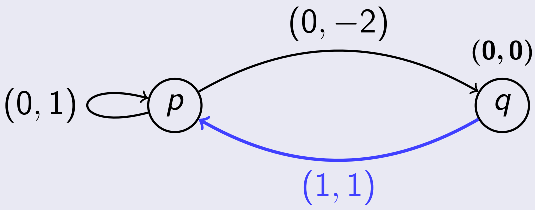

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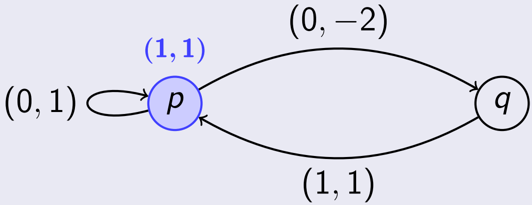

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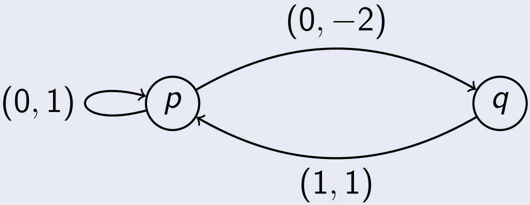
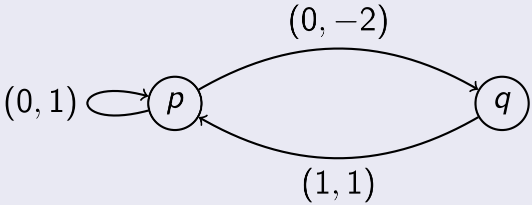


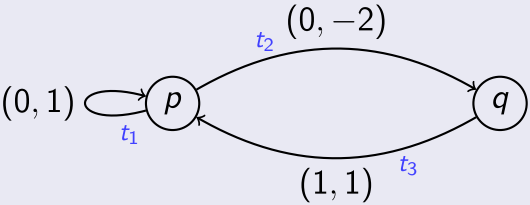
We write $p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v})$ if $\exists$ run from $p(\mathbf{u})$ to $q(\mathbf{v})$

Runs



E.g. $p(0,0) \xrightarrow{*} p(1,1)$

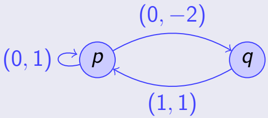
Runs



E.g. $p(0,0) \xrightarrow{t_1\,t_1\,t_2\,t_3} p(1,1)$

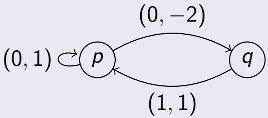
Reachability problem

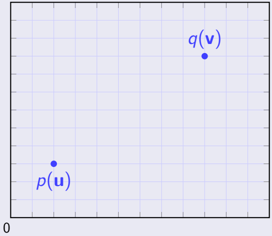
Input: d-VASS V



Reachability problem

Input: d-VASS V and $p(\mathbf{u}), q(\mathbf{v}) \in Q \times \mathbb{N}^d$

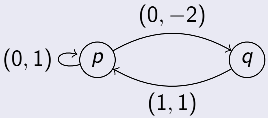


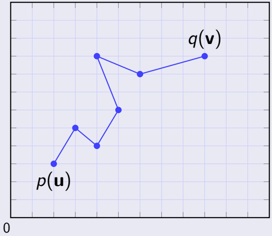


Reachability problem

Input: d-VASS V and $p(\mathbf{u}), q(\mathbf{v}) \in Q \times \mathbb{N}^d$

Question: $p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v})$?





What is known?

VASS **2-VASS**

What is known?

VASS		2-VASS
	'15	
	'13	
	'12	
	'11	
	'09	
	'04	
	'92	
	'86	
	'82	
	'81	
	'79	
EXPSPACE-hard (Lipton)	'76	

What is known?

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	'12	
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	'09	
	'04	
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	'86	
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	'81	
	'79	Decidable (Hopcroft & Pansiot)
EXPSPACE-hard (Lipton)	'76	$\{\mathbf{v} : p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v})\}$ semilinear

What is known?

VASS | **2-VASS**

'15

'13

'12

'11

'09

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'92

'86

'82

'81 — Decidable (Mayr)

'79 — 2-VASS: Decidable (Hopcroft & Pansiot): $\{\mathbf{v} : p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v})\}$ semilinear

'76 — EXPSPACE-hard (Lipton)

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	'13	
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	'86	
Decidable (Kosaraju)	'82	
Decidable (Mayr)	'81	
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	'12	
	'11	
	'09	
	'04	
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	'86	2-EXPTIME (Howell, Huynh, Rosier & Yen)
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	'11	
	'09	
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	'12	
Decidable (Leroux)	'11	
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What is known?

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$\mathbf{F}_{\omega^3}$ (Leroux & Schmitz)	'15	
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	'12	
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	'09	
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What is known?

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	'13	PSPACE-hard (Fearnley & Jurdziński)
	'12	
Decidable (Leroux)	'11	
	'09	
	'04	$\{(\mathbf{u},\mathbf{v}) : p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v})\}$ semilinear (Leroux & Sutre)
Decidable (Lambert)	'92	
	'86	2-EXPTIME (Howell, Huynh, Rosier & Yen)
Decidable (Kosaraju)	'82	
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EXPSPACE-hard (Lipton)	'76	

Theorem

$\exists\, c \;\; \forall\, 2\text{-VASS } V$

$$p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v}) \implies p(\mathbf{u}) \xrightarrow{\pi} q(\mathbf{v}) \;\text{ s.t. }\; |\pi| \leq c^{|V|}$$

Theorem

$\exists\, c\ \ \forall\, \text{2-VASS}\ V$

$$p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v}) \implies p(\mathbf{u}) \xrightarrow{\pi} q(\mathbf{v}) \ \text{ s.t. } \ |\pi| \leq c^{|V|}$$

Corollary

Reachability for 2-VASS $\in$ PSPACE

Theorem

$\exists\, c \;\; \forall\,$ 2-VASS V

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Corollary

Exp. length runs $\implies$ exp. intermediate counter values

Theorem

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Corollary

Exp. length runs $\implies$ exp. intermediate counter values

$\implies$ poly. size intermediate counter values

Theorem

$\exists\, c \;\; \forall$ 2-VASS V

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Corollary

Exp. length runs $\implies$ exp. intermediate counter values

$\implies$ poly. size intermediate counter values

$\implies$ guess run on the fly

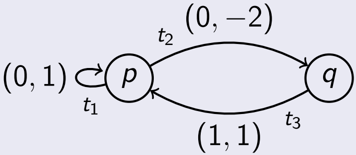
Theorem

$\exists\, c \ \ \forall\, \text{2-VASS } V$

$$p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v}) \implies p(\mathbf{u}) \xrightarrow{\pi} q(\mathbf{v}) \ \text{ s.t. } \ |\pi| \leq c^{|V|}$$

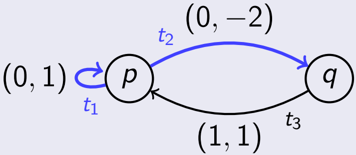
How to prove this theorem?

Linear form of runs



Runs from p to q:

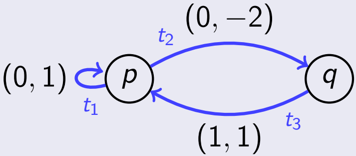
Linear form of runs



Runs from p to q:

$${t_1}^* t_2$$

Linear form of runs



Runs from p to q:

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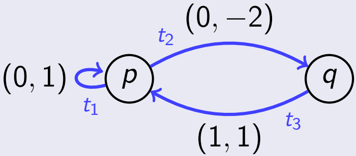
Linear form of runs



Runs from p to q:

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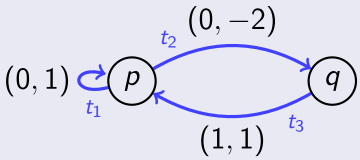
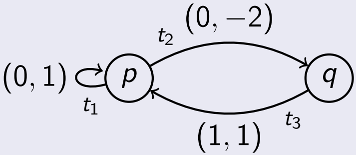
Linear form of runs



Runs from p to q:

$$t_1{}^* t_2 \;\; t_3 t_1{}^* t_2 \;\; t_3 t_1{}^* t_2 \;\; \cdots \;\; t_3 t_1{}^* t_2$$

Linear form of runs



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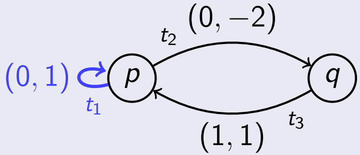

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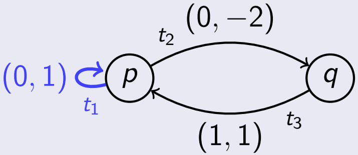
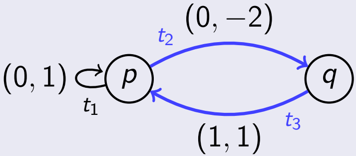


Runs from p to q:

$$t_1{}^* t_2 \; t_3 \quad t_2 \; t_3 \quad t_2 \cdots t_3 \quad t_2$$

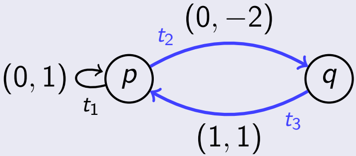
Linear form of runs



Runs from p to q:

$$t_1{}^* t_2 \quad t_3 \quad t_2 \quad t_3 \quad t_2 \cdots t_3 \quad t_2$$

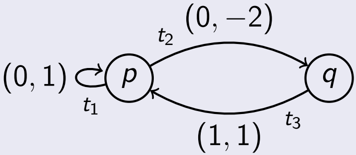
Linear form of runs



Runs from p to q:

$$t_1{}^* t_2 (t_3 \;\; t_2)^*$$

Linear form of runs



Runs from p to q:

$$t_1{}^* t_2 (t_3 \;\; t_2)^*$$

Linear forms always exist in 2-VASS (Leroux & Sutre CONCUR'04)

$$\exists S = \bigcup_{\text{finite}} \alpha_0 {\beta_1}^* \alpha_1 \cdots {\beta_k}^* \alpha_k$$

Linear forms always exist in 2-VASS (Leroux & Sutre CONCUR'04)

$$\exists S = \bigcup_{\text{finite}} \underbrace{\alpha_0 {\beta_1}^* \alpha_1 \cdots {\beta_k}^* \alpha_k}_{\textit{linear path scheme}}$$

Linear forms always exist in 2-VASS (Leroux & Sutre CONCUR'04)

$\exists S = \bigcup_{\text{finite}} \alpha_0 {\beta_1}^* \alpha_1 \cdots {\beta_k}^* \alpha_k$ such that

$$p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v}) \implies p(\mathbf{u}) \xrightarrow{\pi \in S} q(\mathbf{v})$$

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<u>Small</u> linear forms in 2-VASS (B., Finkel, Göller, Haase & McKenzie LICS'15)

- $|\alpha_i|, |\beta_i| \quad \leq (|Q| + \|T\|)^{O(1)}$

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- $|\alpha_i|, |\beta_i| \quad \leq (|Q| + \|T\|)^{O(1)}$
- $k \quad \in O(|Q|^2)$

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- $|\alpha_i|, |\beta_i| \leq (|Q| + \|T\|)^{O(1)}$
- $k \in O(|Q|^2)$
- $*$-exponents $\leq (|Q| + \|T\| + \|\mathbf{u}\| + \|\mathbf{v}\|)^{O(1)}$

Linear forms always exist in 2-VASS (Leroux & Sutre CONCUR'04)

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$$p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v}) \implies p(\mathbf{u}) \xrightarrow{\pi \in S} q(\mathbf{v})$$

Small linear forms in 2-VASS (B., Finkel, Göller, Haase & McKenzie LICS'15)

- $|\alpha_i|, |\beta_i| \leq$ exponential
- $k \in$ polynomial
- $*$-exponents $\leq$ exponential

Open questions

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- 3-VASS: PSPACE-hard and $\in \mathbf{F}_{\omega^3}$. Better bounds?

Thank you! Merci!