Logics for Continuous Reachability in Petri Nets and Vector Addition Systems with States

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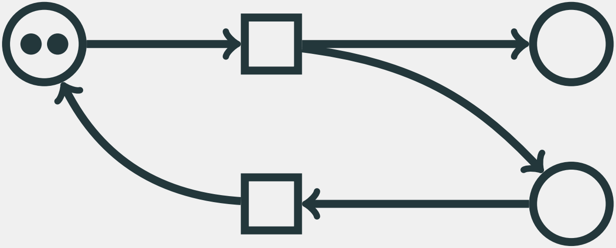
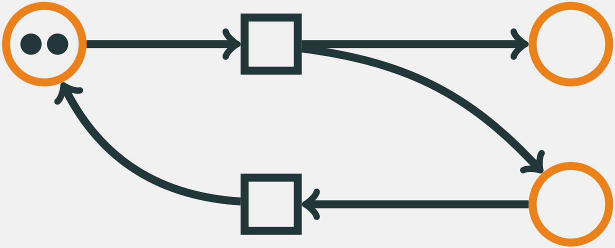
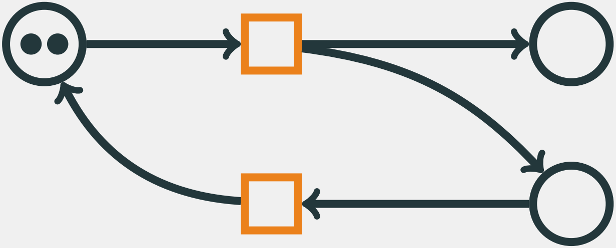
University of Oxford

Petri nets

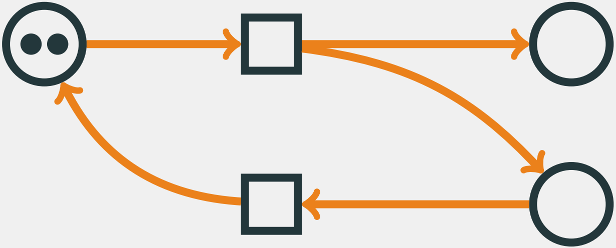




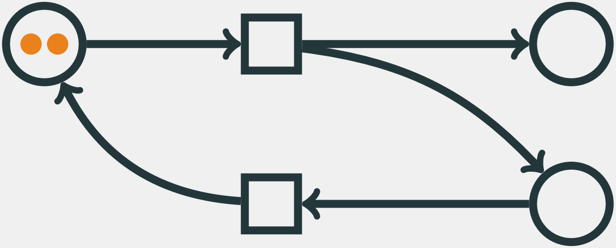
Places



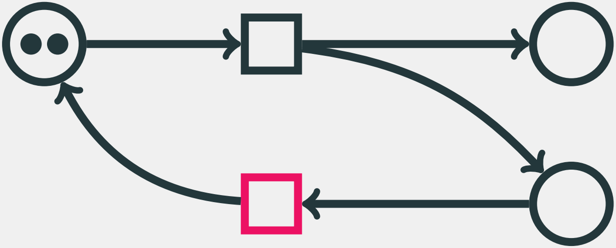
Transitions



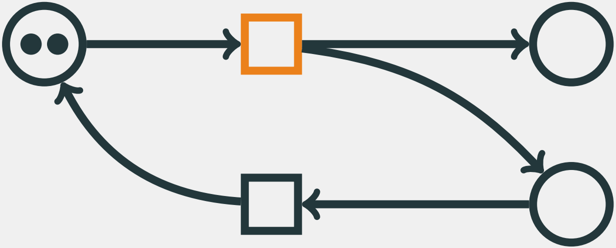
Arcs



Tokens

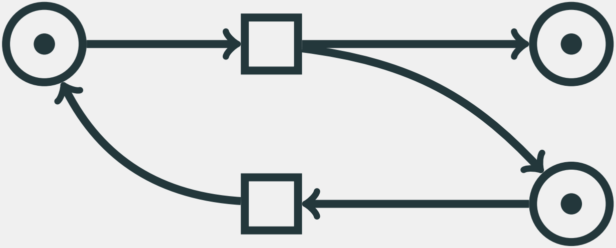


Disabled

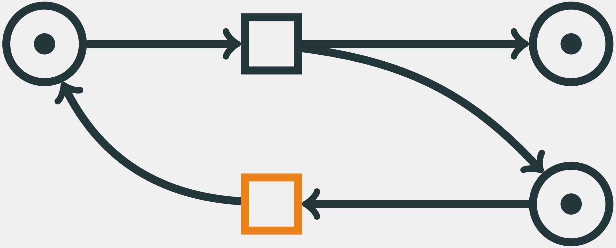


Enabled

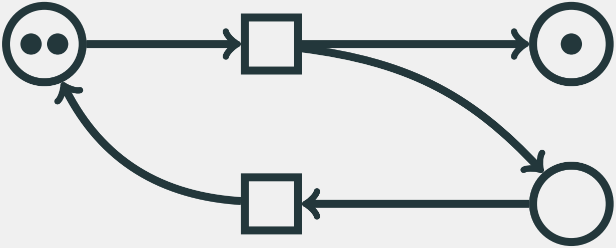
Petri nets

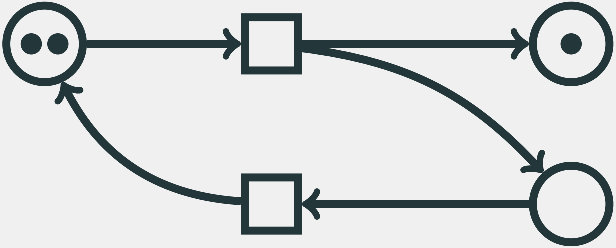


Petri nets

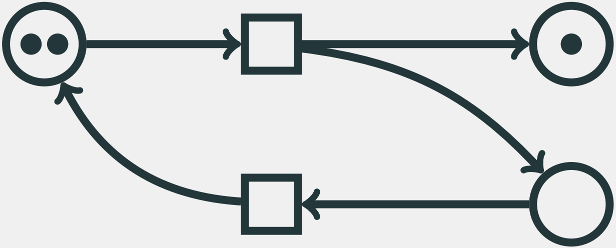


Petri nets





$$(2, 0, 0) \xrightarrow{*}_{\mathbb{N}} (2, 1, 0)$$



Reachability: $\boldsymbol{u} \xrightarrow{*}_{\mathbb{N}} \boldsymbol{v}$?

Petri nets

Concurrent programs

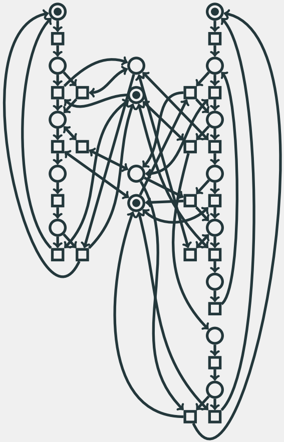
Protocols

Business processes

Biological processes

⋮

correct ? ⟶

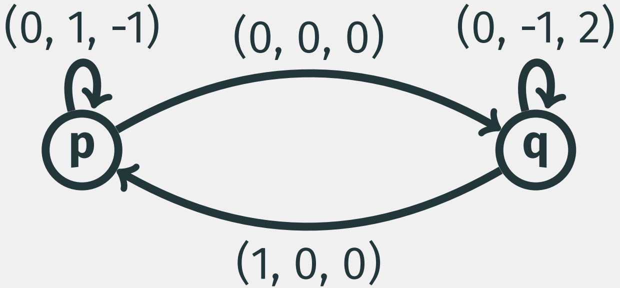


$$\text{Reachability: } \boldsymbol{u} \xrightarrow{*}_{\mathbb{N}} \boldsymbol{v}\,?$$

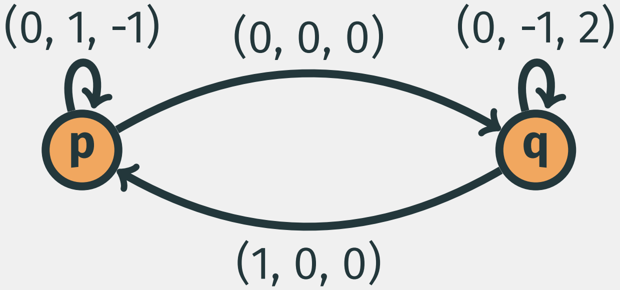
Verifying multi-threaded programs

Counters: number of threads

States: shared resources

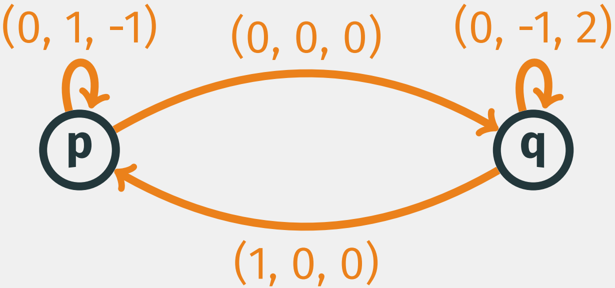

(0, 1, -1)
(0, 0, 0)
(0, -1, 2)
p
q
(1, 0, 0)

Petri nets / Vector addition systems with states (VASS)



Control-states

Petri nets / Vector addition systems with states (VASS)



Transitions

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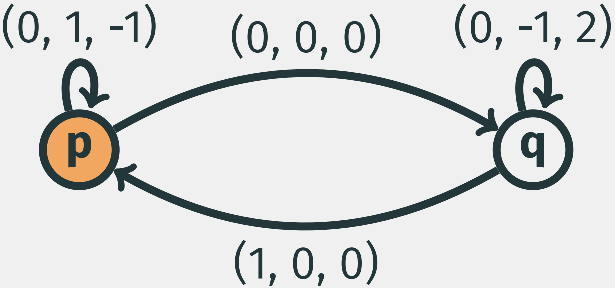
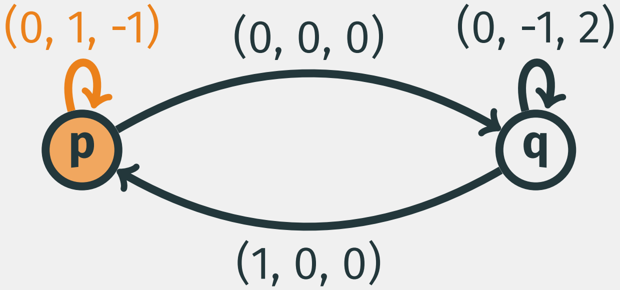


p(0, 0, 1)

Petri nets / Vector addition systems with states (VASS)



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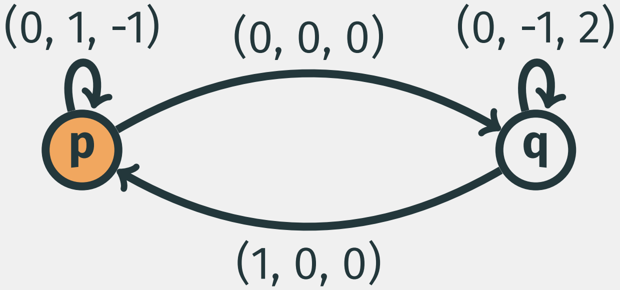
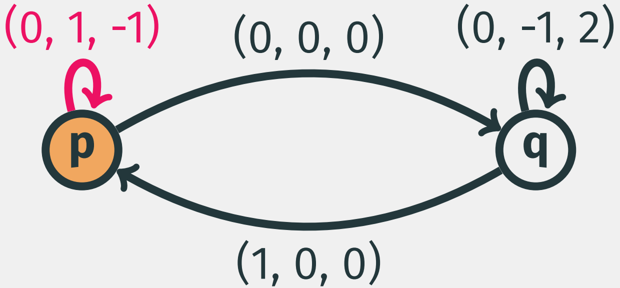


p(0, 1, 0)

Petri nets / Vector addition systems with states (VASS)



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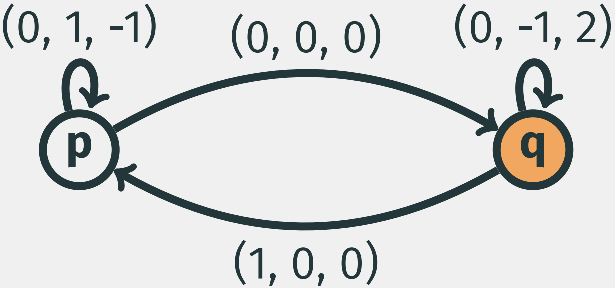
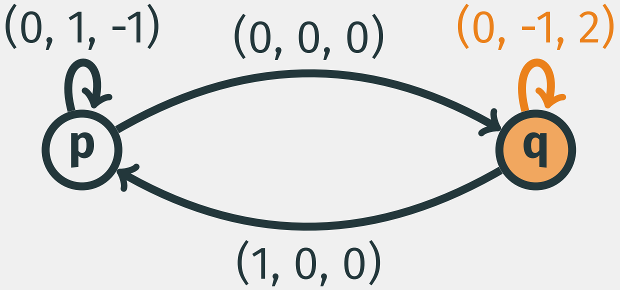
Petri nets / Vector addition systems with states (VASS)



q(0, 1, 0)

Petri nets / Vector addition systems with states (VASS)



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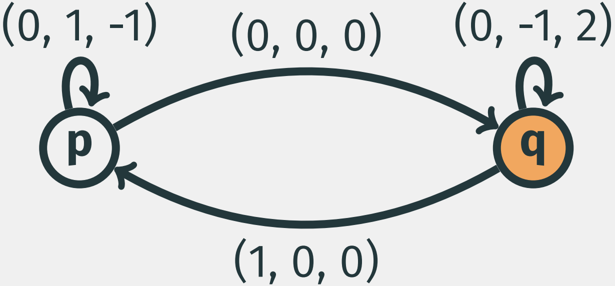
Petri nets / Vector addition systems with states (VASS)



q(0, 0, 2)

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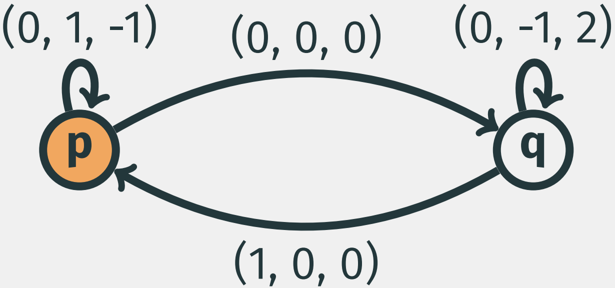
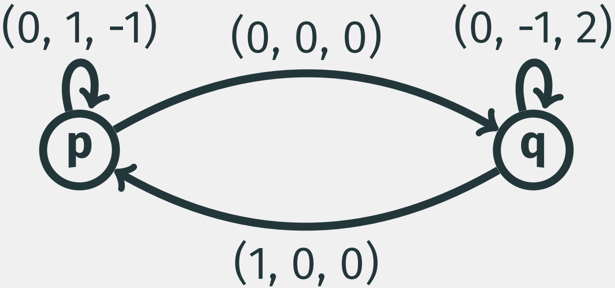
Petri nets / Vector addition systems with states (VASS)

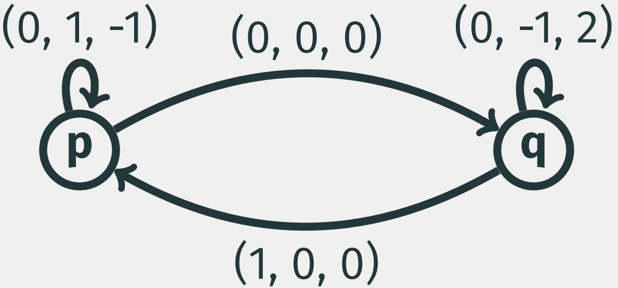


p(1, 0, 2)

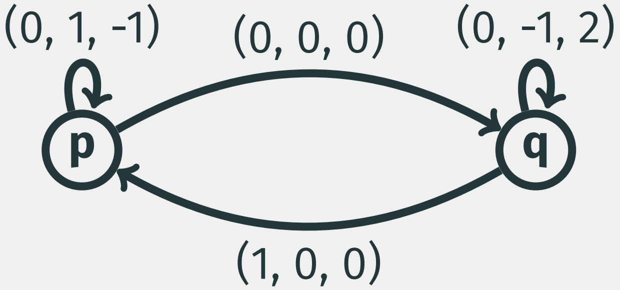
Petri nets / Vector addition systems with states (VASS)



$$p(0, 0, 1) \xrightarrow{*}_{\mathbb{N}} p(1, 0, 2)$$



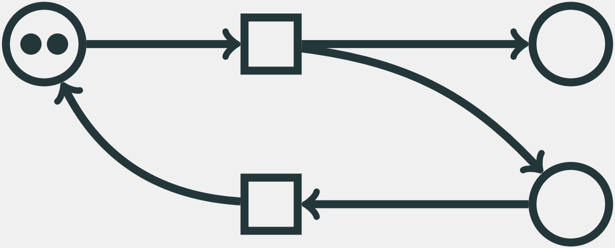
$$p(0,0,1) \xrightarrow{*}_{\mathbb{N}} p(x,y,z) \iff 0 < y + z \leq 2^x$$



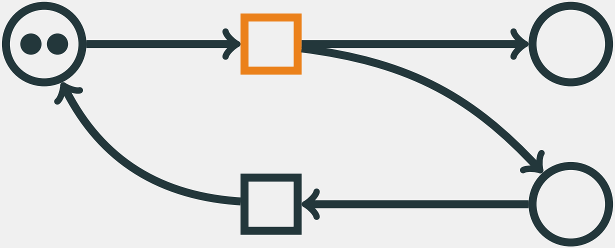
Reachability: $p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{N}} q(\boldsymbol{v})$?

Reachability is...

- equivalent for Petri nets and VASS
- not expressible in FO($\mathbb{N}$, +, <)
- EXPSPACE-hard (Lipton '76)
- solvable in cubic-Ackermannian time
(Leroux and Schmitz LICS'15)



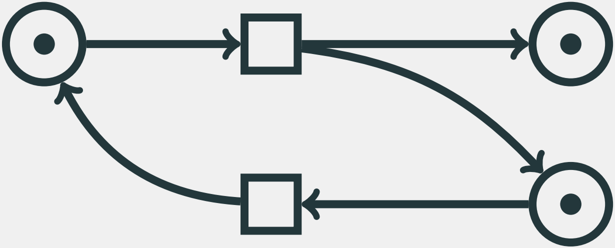
Can fire transitions fractionally



Can fire transitions fractionally

Continuous Petri nets

David & Alla '87



Can fire transitions
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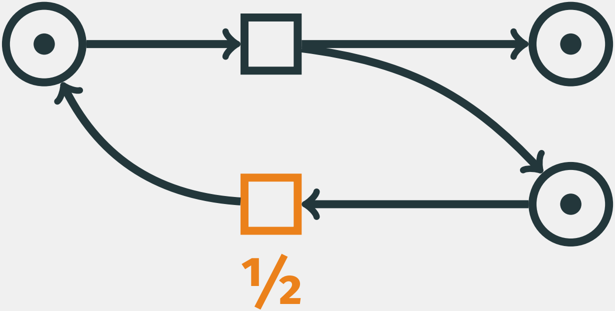

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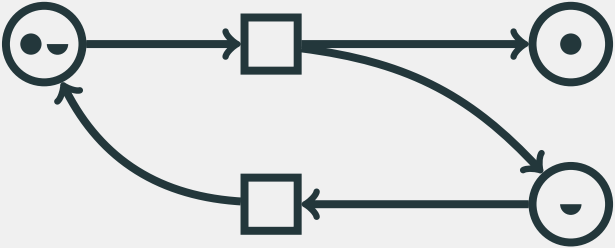

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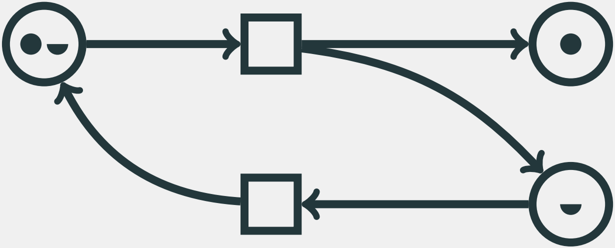
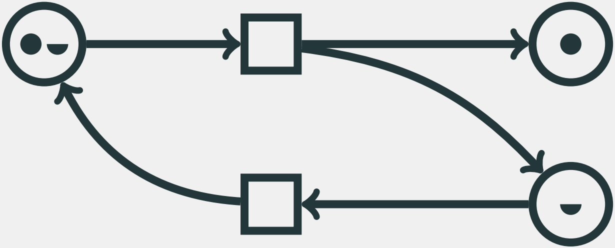


$$(2, 0, 0) \xrightarrow{*}_{\mathbb{Q}_+} (1\tfrac{1}{2}, 1, \tfrac{1}{2})$$



Continuous reachability: $\boldsymbol{u} \xrightarrow{*}_{\mathbb{Q}_+} \boldsymbol{v}$?

Continuous reachability is...

- an over-approximation:

$$\neg(\boldsymbol{u} \xrightarrow{*}_{\mathbb{Q}_+} \boldsymbol{v}) \text{ implies } \neg(\boldsymbol{u} \xrightarrow{*}_{\mathbb{N}} \boldsymbol{v})$$

- PTIME-complete (Fraca and Haddad PN'13)

- expressible in $\exists$ FO($\mathbb{Q}_+$, +, <)

(B., Finkel, Haase and Haddad TACAS'16)

Continuous Petri nets

David & Alla '87

Continuous reachability is...

- an over-approximation

fast with SMT solver
...but NP-complete

- PTIME-complete
- expressible in $\exists$ FO($\mathbb{Q}_+$, +, <)

Continuous Petri nets

David & Alla '87

Continuous reachability is...

- an over-approximation

often good

...but no control-states

- PTIME-complete
- expressible in $\exists\,\mathrm{FO}(\mathbb{Q}_+, +, <)$

new fragment of $\exists$ FO($\mathbb{Q}_+$, +, <)

- PTIME-complete
- equivalent to Petri net continuous reachability

new model: continuous VASS

- with control-states
- with reachability equivalent to $\exists$ FO($\mathbb{Q}$, +, <)

new fragment of $\exists$ FO($\mathbb{Q}_+$, +, <)

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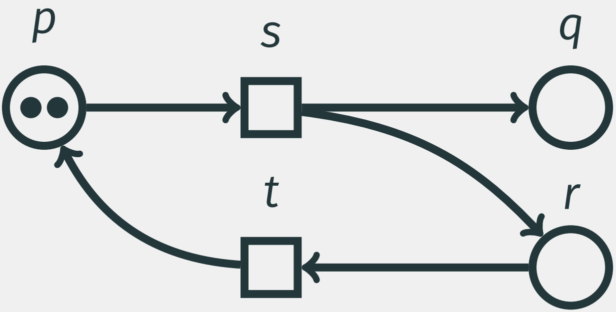
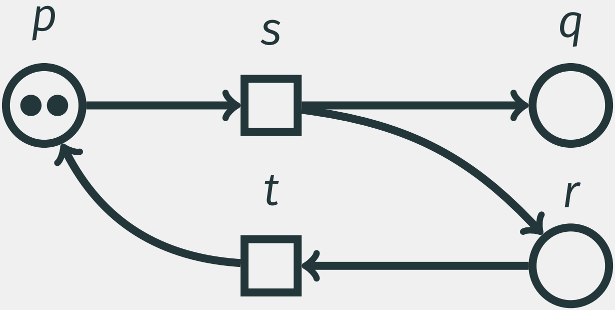
Expressing continuous reachability



$\boldsymbol{y}$ reachable $\iff$ $\exists$ Parikh vector π s.t.

Expressing continuous reachability

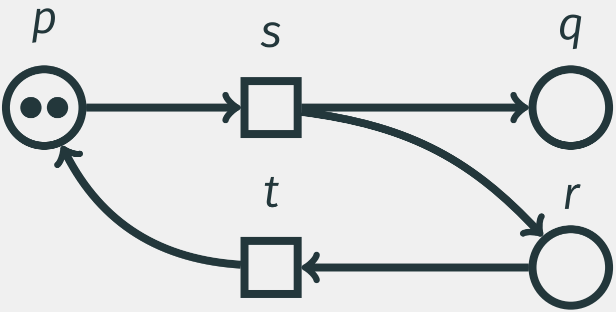


$\boldsymbol{y}$ reachable $\iff$ $\exists$ Parikh vector π s.t.

a) $\boldsymbol{y}(p) = 2 - \pi(s) + \pi(t)$ $\quad \boldsymbol{y}(q) = \pi(s)$

$\boldsymbol{y}(r) = \pi(s) + \pi(t)$

Expressing continuous reachability

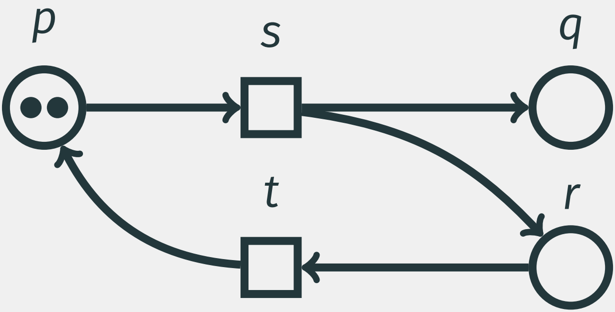


$\boldsymbol{y}$ reachable $\iff \exists$ Parikh vector π s.t.

b) $\{u : \pi(u) > 0\}$ firable from (2, 0, 0)

c) $\{u : \pi(u) > 0\}$ firable to $\boldsymbol{y}$

Expressing continuous reachability



linear size $\exists FO(\mathbb{Q}_+, +, <)$ formula

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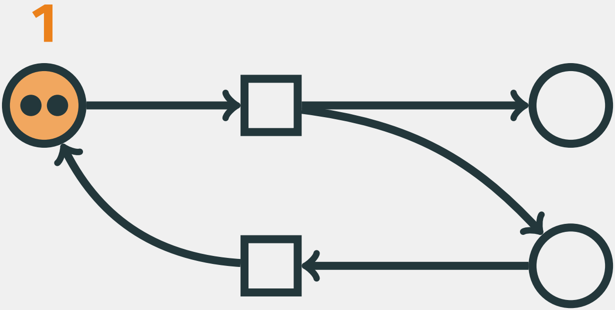

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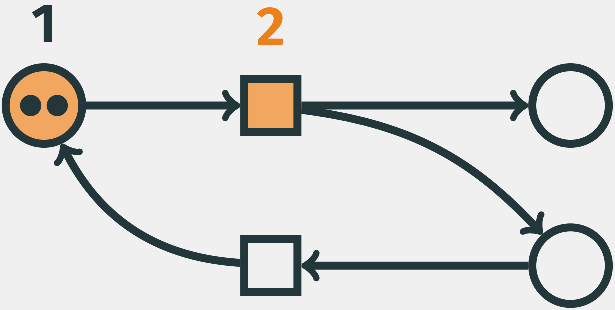

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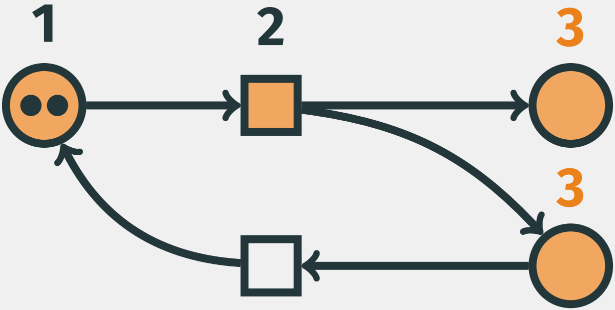

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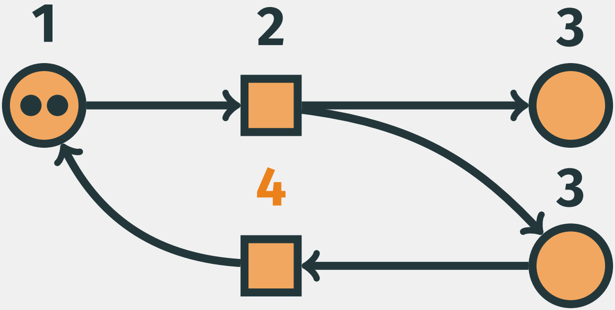

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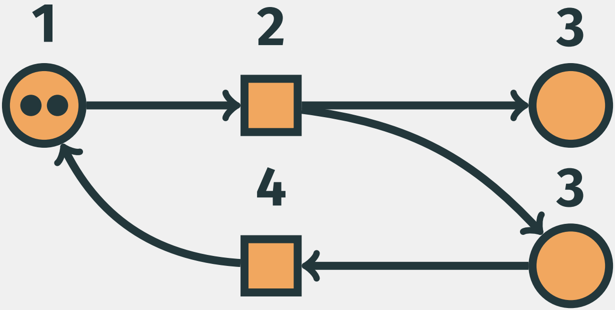

linear size $\exists FO(\mathbb{Q}_+, +, <)$ formula

NP-complete

Conjunction of terms of the form

$$a_1 \cdot x_1 + \ldots + a_n \cdot x_n \sim c \vee \bigvee_{1 \leq i \leq m} \bigwedge_{j \in J_i} x_j > 0$$

Conjunction of terms of the form

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variables over $\mathbb{Q}_+$

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coeff. in $\mathbb{Z}$

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variables over $\mathbb{Q}_+$

coeff. in $\mathbb{Z}$

$\sim$ is $\geq$ or $>$

Examples :

$$x = 0 \quad \equiv \quad -x \geq 0$$

Convex linear Horn constraints

Examples :

$$x = 0 \quad \equiv \quad -x \geq 0$$

$$x > 0 \rightarrow \bigvee_{y \in Y} y > 0 \quad \equiv \quad x = 0 \vee \bigvee_{y \in Y} y > 0$$

Examples :

$$x = 0 \quad \equiv \quad -x \geq 0$$

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$$x > 0 \leftarrow \bigvee_{y \in Y} y > 0 \quad \equiv \quad \bigwedge_{y \in Y} (y = 0 \vee x > 0)$$

$$x > 0 \leftarrow \bigwedge_{y \in Y} y > 0 \quad \equiv \quad \text{not expressible}$$

Examples :

- graph reachability
- strong connectivity (from a node)

in subgraph $G[u : \boldsymbol{x}_u > 0]$

Convex linear Horn constraints

Theorem

Continuous Petri net reachability is expressible by a quadratic size formula

Proof idea

$$\boldsymbol{x} + \boldsymbol{C} \cdot \pi = \boldsymbol{y} \ \wedge \ \text{firability constraints}$$

Convex linear Horn constraints

Theorem

Satisfiability is PTIME-complete

Proof idea

- $\in$ PTIME: algorithm exploiting convexity
- Hardness: from linear prog. feasability

Related work:

CSP of linear Horn constraints
(Jonsson and Backström '98
Koubarakis '01)

Convex linear Horn constraints

incomparable

Related work:

CSP of linear Horn constraints

(Jonsson and Backström '98

Koubarakis '01)

Solutions as reachability sets

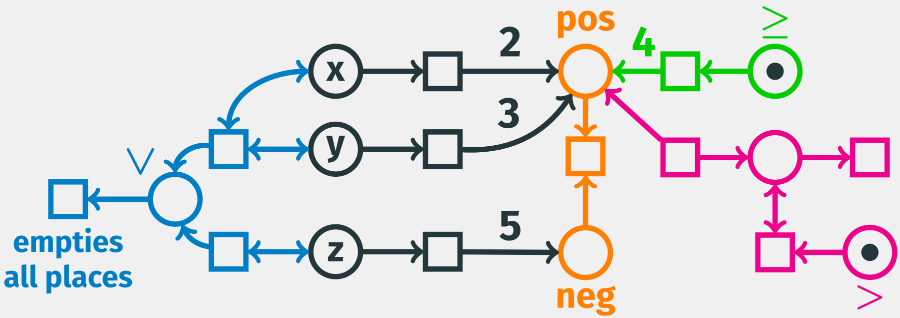
Theorem

Convex linear Horn constraints solutions
are expressible as
reachability sets of continuous Petri nets

$$2x + 3y - 5z > -4 \lor (x > 0 \land y > 0) \lor z > 0$$

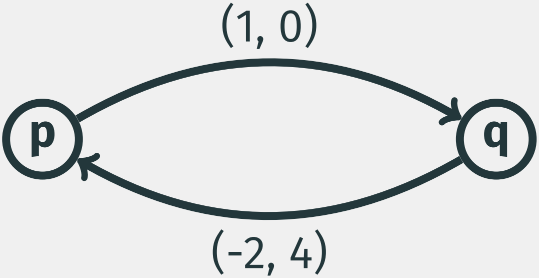
$$\iff$$

all places can be emptied



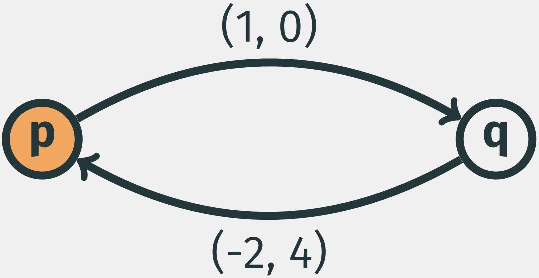
Continuous VASS:

- Hybrid model
- subsume continuous Petri nets



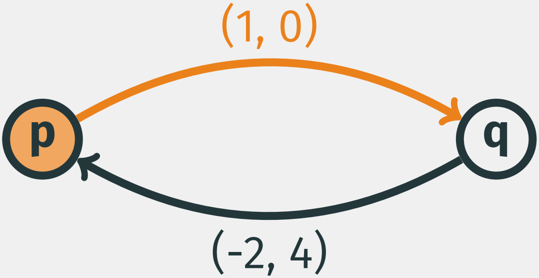
Transitions can be scaled by $0 < x \leq 1$

Continuous VASS



p(0, 0)

Continuous VASS



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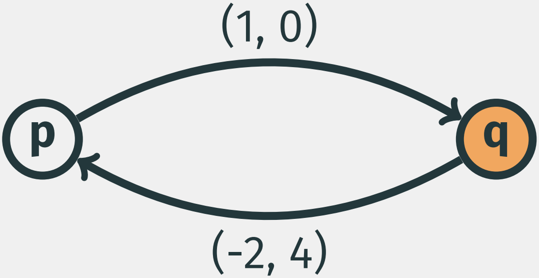
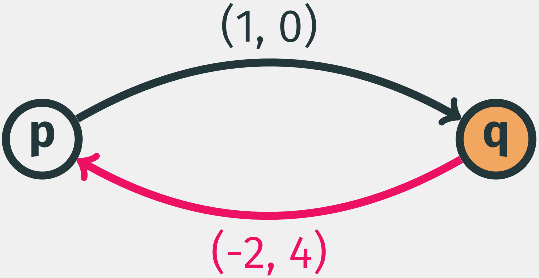


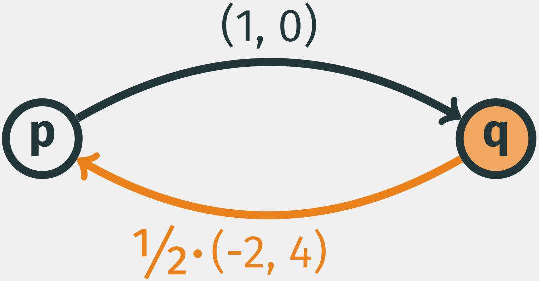
q(1, 0)

Continuous VASS



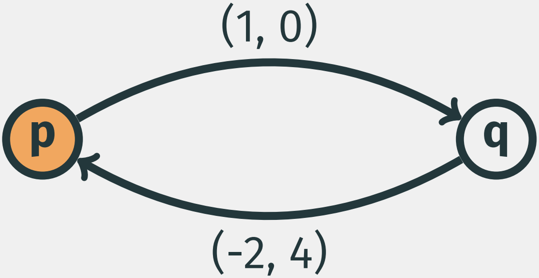
q(1, 0)

Continuous VASS



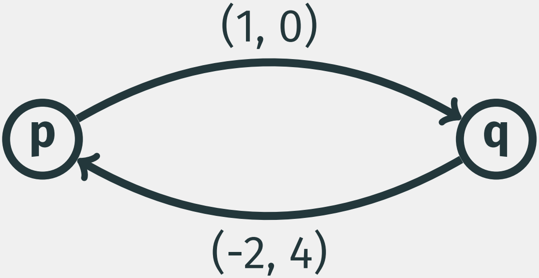
q(1, 0)

Continuous VASS

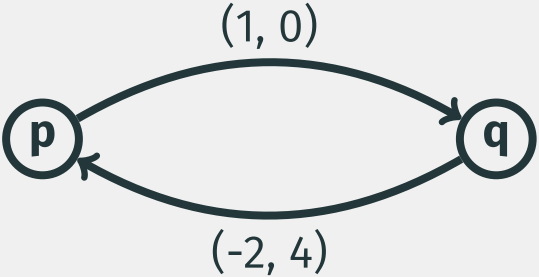


p(0, 2)

Continuous VASS



$$p(0,0) \xrightarrow{*}_{\mathbb{Q}_+} q(0,2)$$



Reachability: $p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{Q}_+} q(\boldsymbol{v})$?

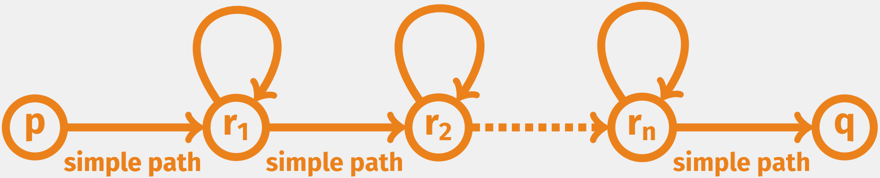
Continuous VASS: logical characterization

Theorem

$$p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{Q}_+} q(\boldsymbol{v})$$

is expressible in $\exists\,\mathrm{FO}(\mathbb{Q}, +, <)$

Continuous VASS: logical characterization



Theorem

$$p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{Q}_+} q(\boldsymbol{v})$$

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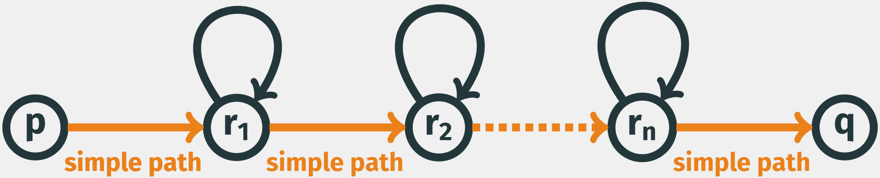
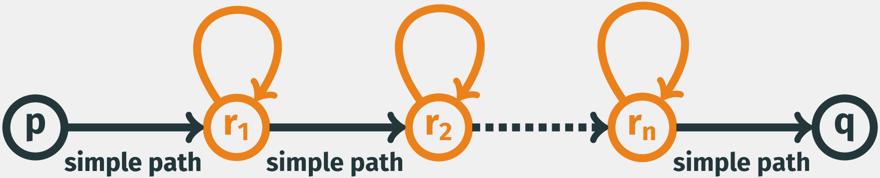
Continuous VASS: logical characterization



Fixed length runs are easy to express in $\exists FO(\mathbb{Q}, +, <)$

Continuous VASS: logical characterization



$$r(\boldsymbol{x}) \xrightarrow{*}_{\mathbb{Q}_+} r(\boldsymbol{y})$$

$$r(\boldsymbol{x}) \xrightarrow{*}_{\mathbb{Q}_+} r(\boldsymbol{y})$$

Equivalent $\updownarrow$

$$r(\boldsymbol{x}) \xrightarrow{*}_{\mathbb{Q}} r(\boldsymbol{y})$$
$$\wedge\ r(\boldsymbol{x}) \xrightarrow{*}_{\mathbb{Q}_+} r(\_)$$
$$\wedge\ r(\_) \xrightarrow{*}_{\mathbb{Q}_+} r(\boldsymbol{y})$$

Continuous VASS: logical characterization

Characterization of admissible paths

$$r(\boldsymbol{x}) \xrightarrow{*}_{\mathbb{Q}} r(\boldsymbol{y})$$

$$\land\ r(\boldsymbol{x}) \xrightarrow{*}_{\mathbb{Q}_+} r(\_)$$

$$\land\ r(\_) \xrightarrow{*}_{\mathbb{Q}_+} r(\boldsymbol{y})$$

Continuous VASS: logical characterization

Next slide

$$r(\boldsymbol{x}) \xrightarrow{*}_{\mathbb{Q}} r(\boldsymbol{y})$$

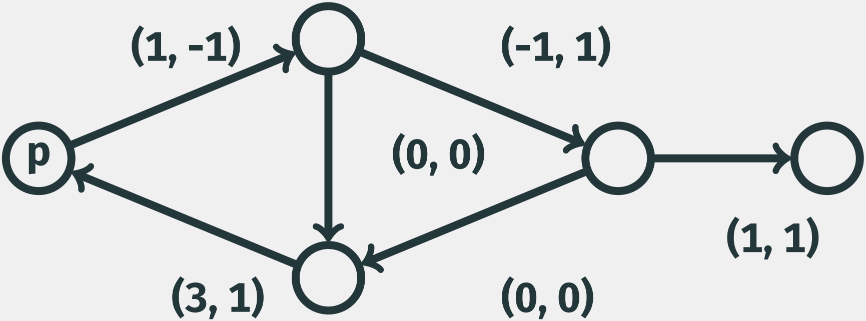
$$\wedge\ r(\boldsymbol{x}) \xrightarrow{*}_{\mathbb{Q}_+} r(\_)$$

$$\wedge\ r(\_) \xrightarrow{*}_{\mathbb{Q}_+} r(\boldsymbol{y})$$

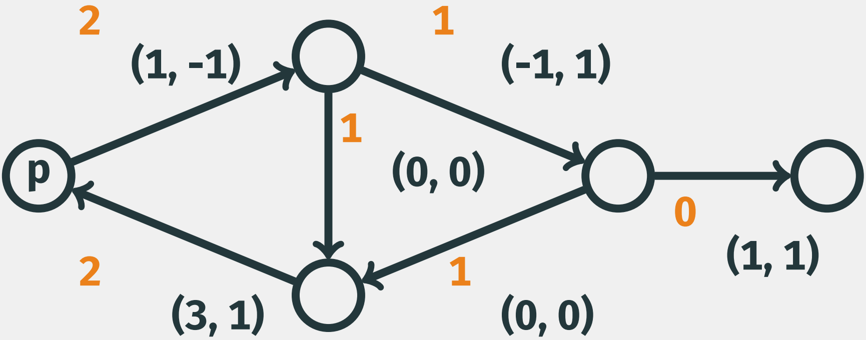
VASS cyclic $\mathbb{Q}$-reachability

$$p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{Q}} p(\boldsymbol{v})?$$

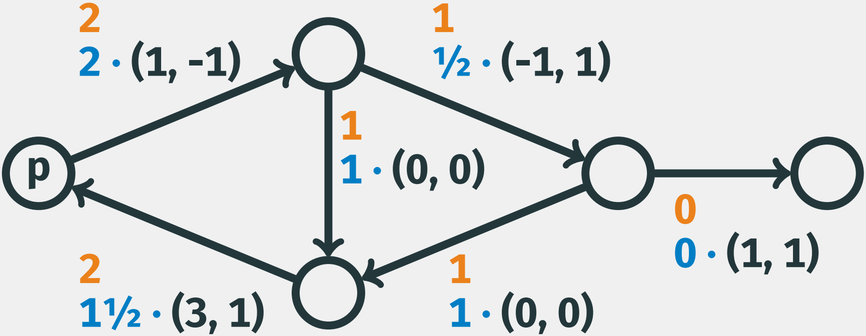
$$p(0,0) \xrightarrow{*}_{\mathbb{Q}} p(6,0)?$$



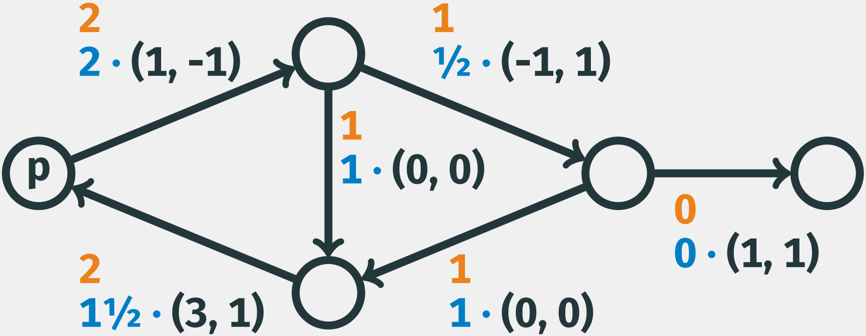
Guess transitions occurrences



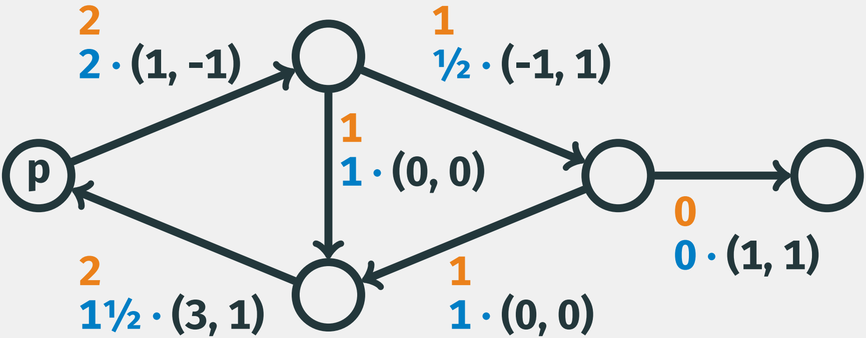
Guess scaling factors



occurrences ≥ scaling

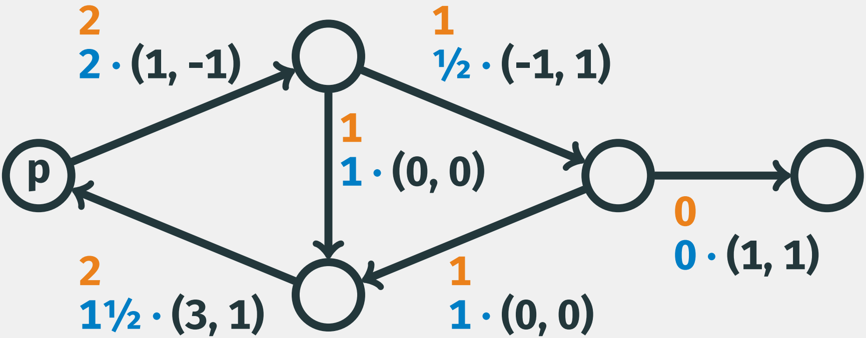


occurrences > 0 $\leftrightarrow$ scaling > 0

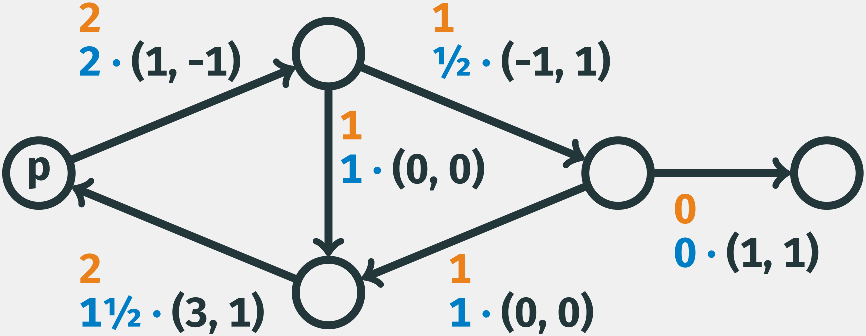


VASS cyclic $\mathbb{Q}$-reachability

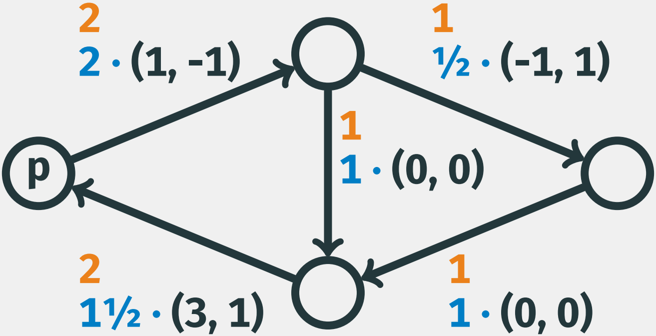
$$(0, 0) + \sum_{t \in T} \boldsymbol{f}_t \cdot \boldsymbol{z}_t = (6, 0)$$



G[$\boldsymbol{x}_t$ > 0] strongly connected

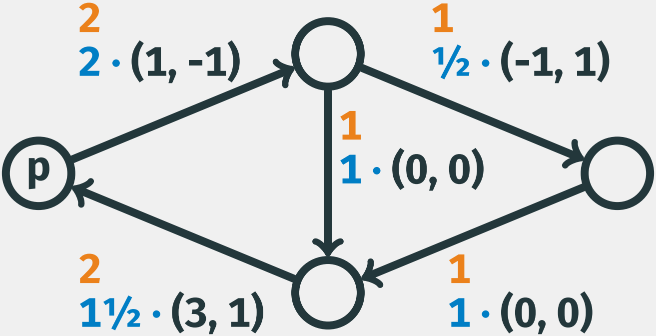


G[x_t > 0] strongly connected



VASS cyclic $\mathbb{Q}$-reachability

$$\sum_{t \in \text{in}(q)} x_t = \sum_{t \in \text{out}(q)} x_t$$



VASS cyclic $\mathbb{Q}$-reachability

Theorem

These conditions are expressible as convex linear Horn constraints

Corollary

VASS cyclic $\mathbb{Q}$-reachability is in PTIME

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VASS cyclic $\mathbb{Q}$-reachability is in PTIME

Conclusion: summary

- new PTIME fragment of $\exists\mathrm{FO}(\mathbb{Q}, +, <)$
- logical characterization of continuous Petri nets
- new hybrid model: continuous VASS
- characterization of $\exists\mathrm{FO}(\mathbb{Q}, +, <)$

Conclusion: future work

- optimization of continuous Petri nets
  w.r.t. to linear objective functions
  (e.g. constant-rate multi-mode systems of Alur *et al.* HSCC'12)

- games on continuous Petri nets
  (e.g. Minkowski games of Le Roux *et al.* STACS'17)

- more succinct encoding of
  continuous VASS reachability

Thank you !

Takk fyrir !