Automatic Analysis of Expected Termination Time for Population Protocols

Michael Blondin

Joint work with Javier Esparza and Antonín Kučera

Technical University of Munich

Overview

Population protocols: distributed computing model for massive networks of passively mobile finite-state agents

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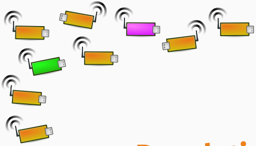

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Can model *e.g.* networks of passively mobile sensors and chemical reaction networks

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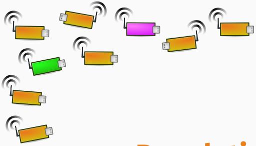

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Protocols compute predicates of the form $\varphi \colon \mathbb{N}^d \to \{0, 1\}$

e.g. if φ is unary, then $\varphi(n)$ is computed by n agents

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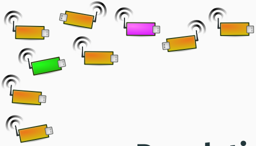

Population protocols: distributed computing model for massive networks of passively mobile finite-state agents

This talk: automatic derivation of upper bounds on the running time of protocols

Population protocols

Angluin *et al.* PODC'04

- anonymous mobile agents with very few resources
- agents change states via random pairwise interactions
- each agent has opinion true/false
- computes by stabilizing agents to some opinion

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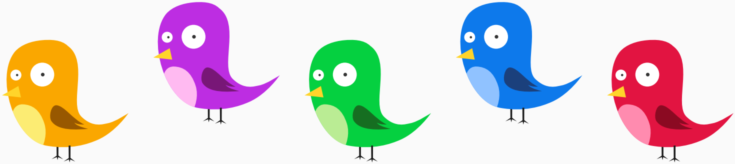

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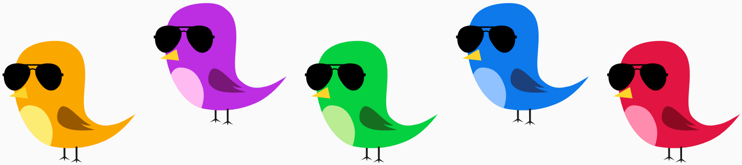

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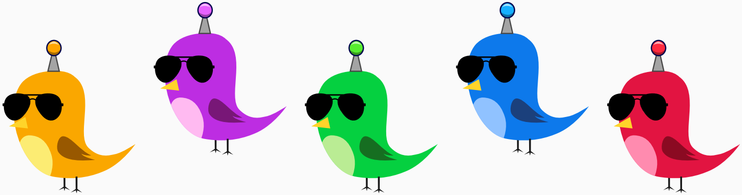

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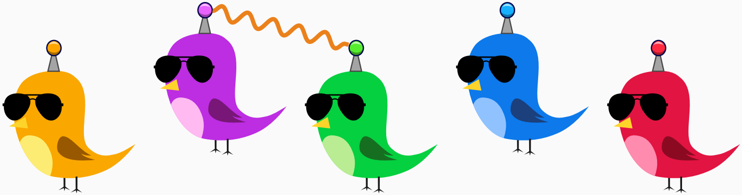

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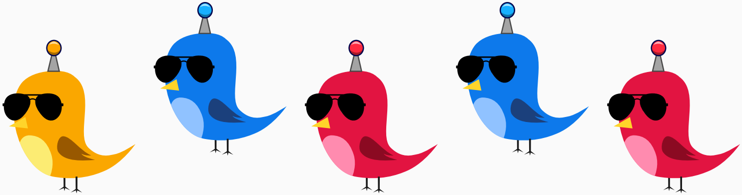

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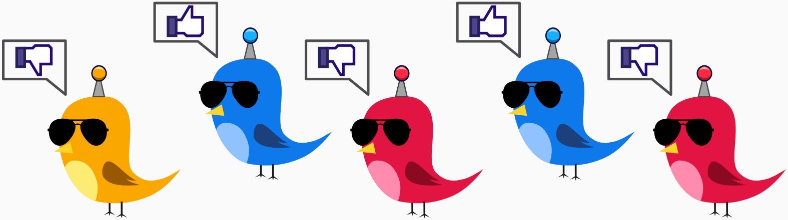

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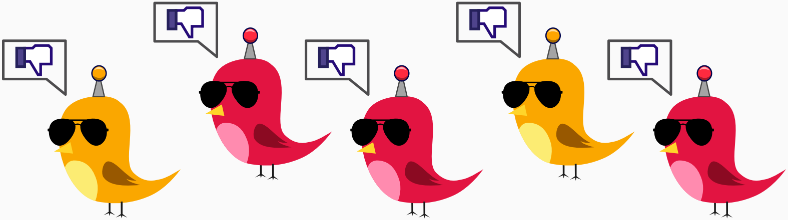

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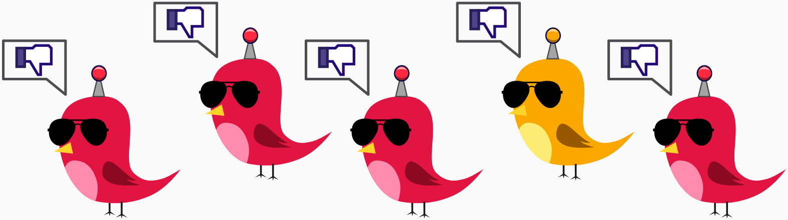

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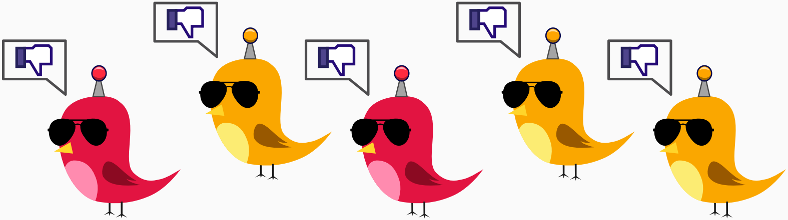
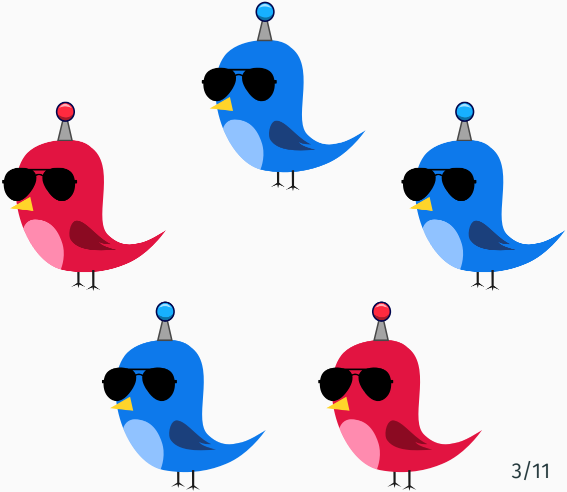


Example: majority protocol

At least as many blue birds than red birds?



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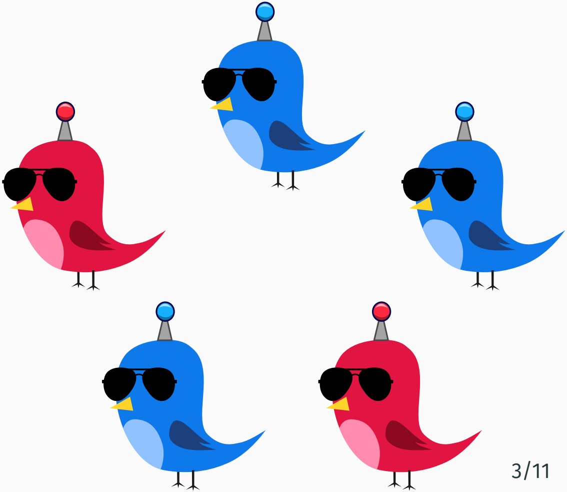

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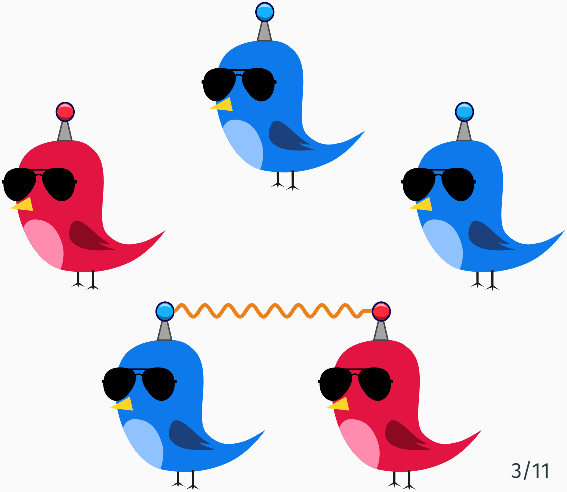

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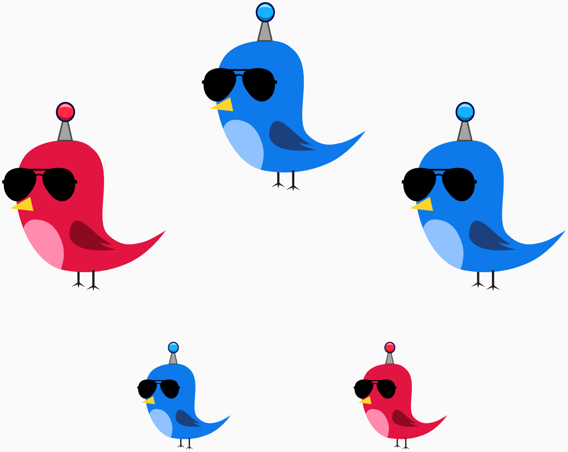

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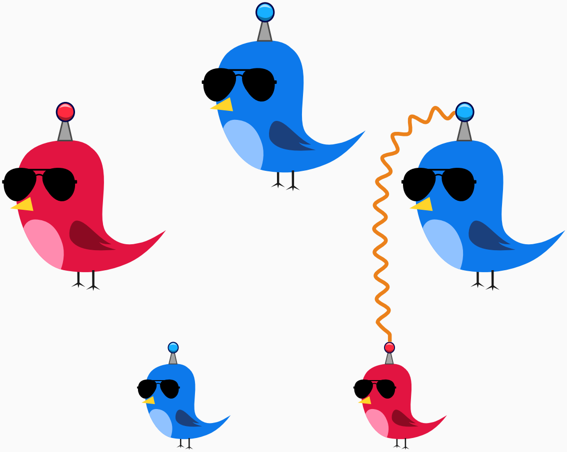

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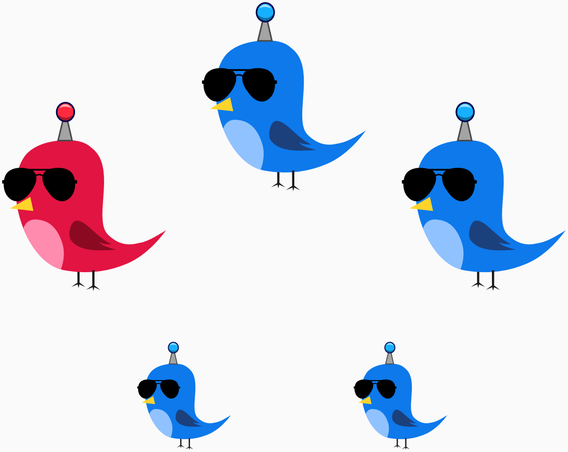

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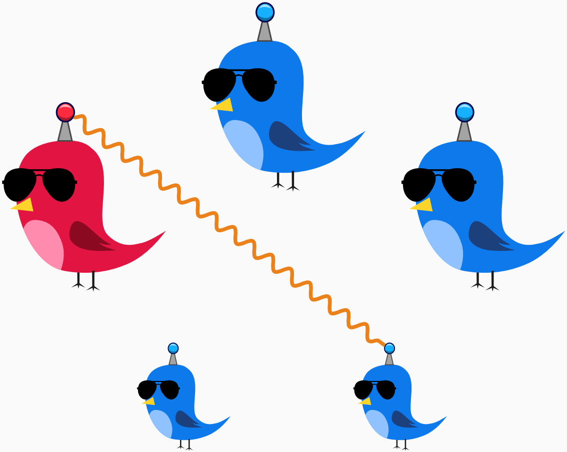

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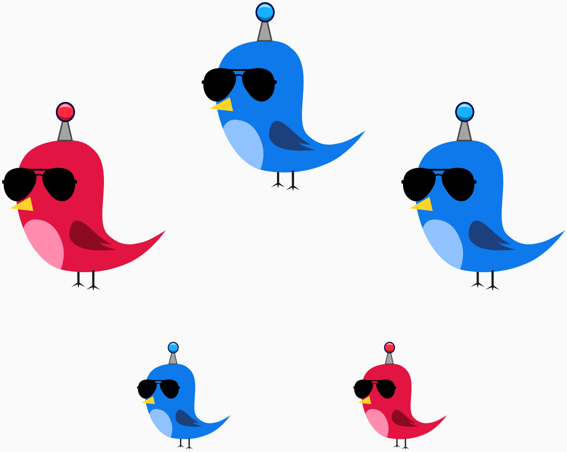

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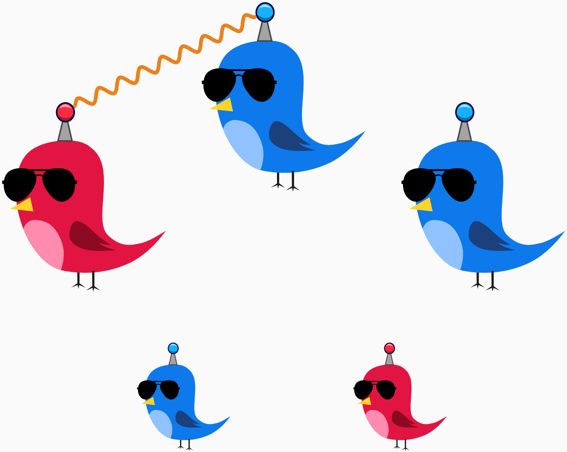

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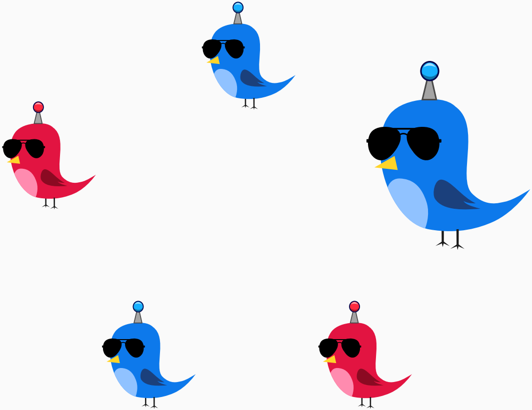

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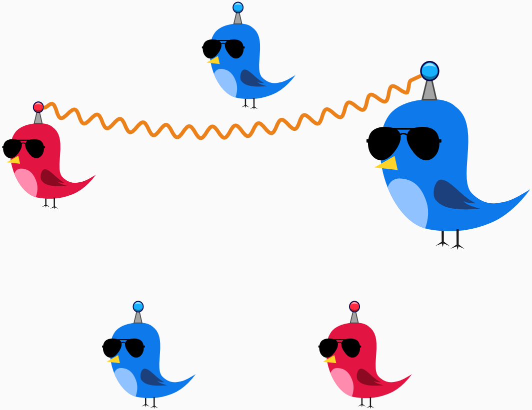

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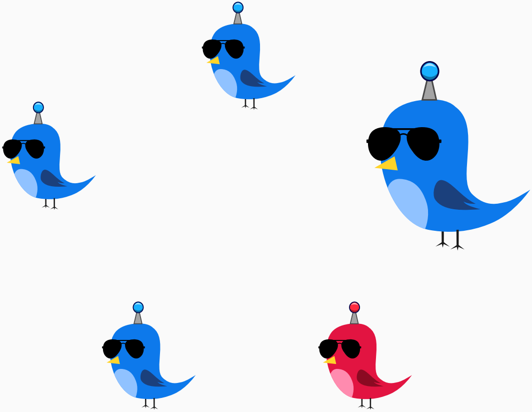

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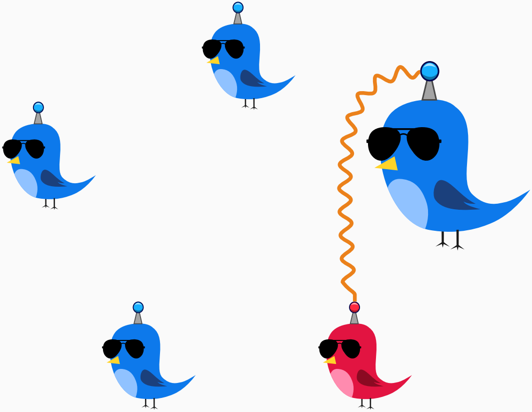

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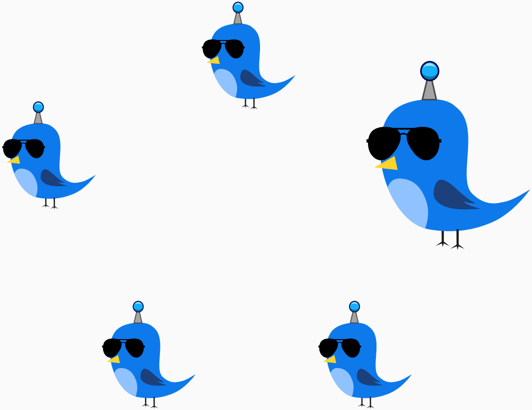

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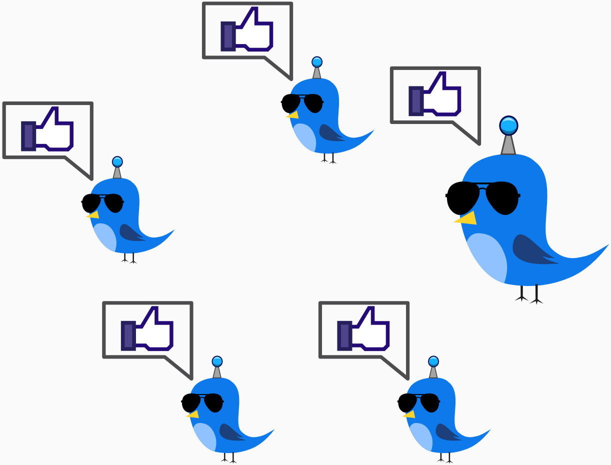

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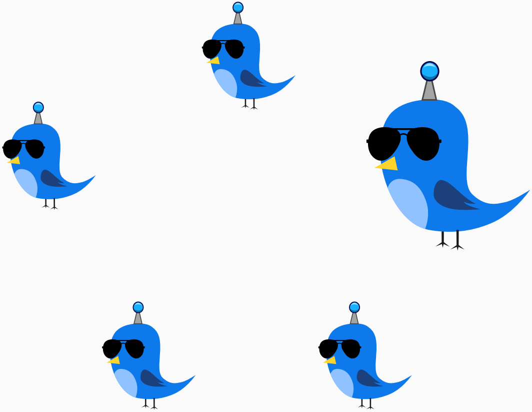
- **To break ties:** small blue birds convert small red birds



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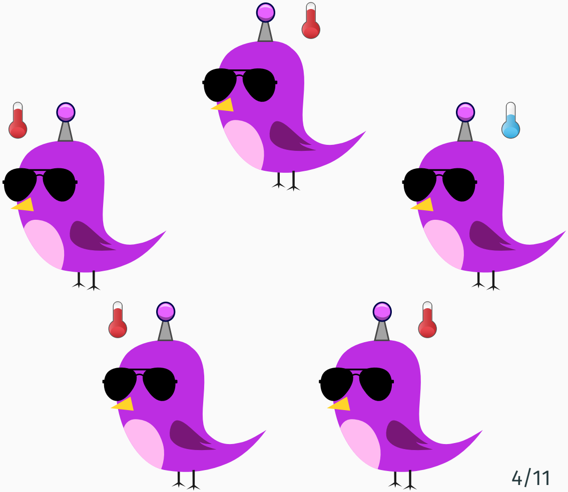

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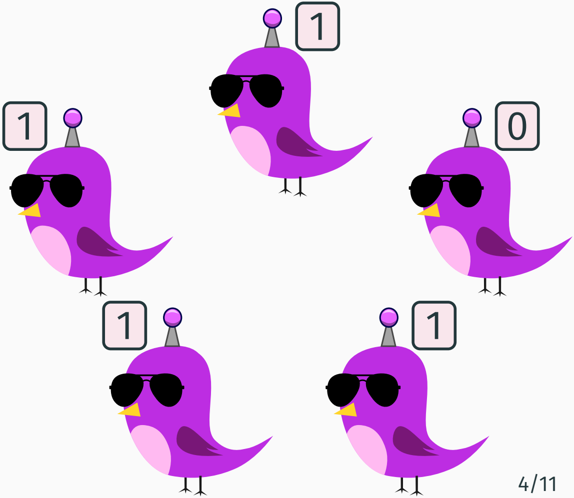

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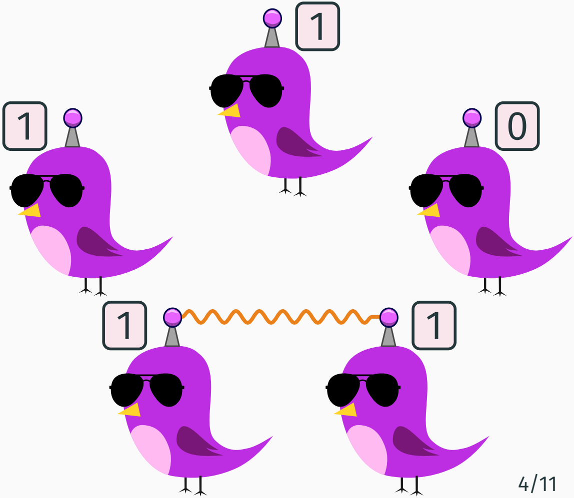

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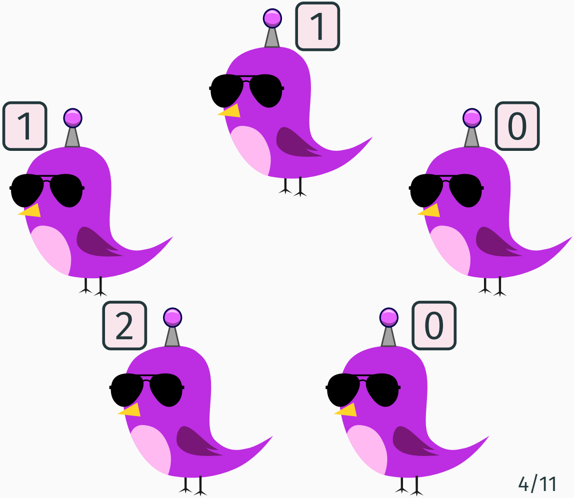

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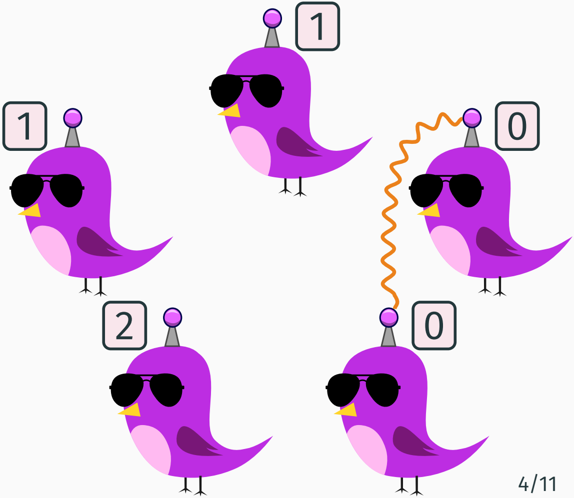

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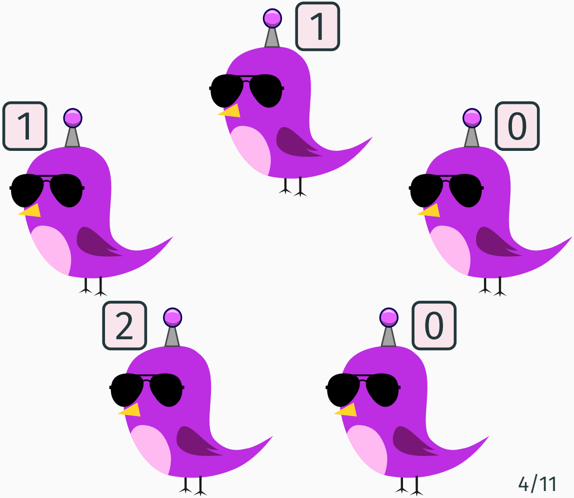

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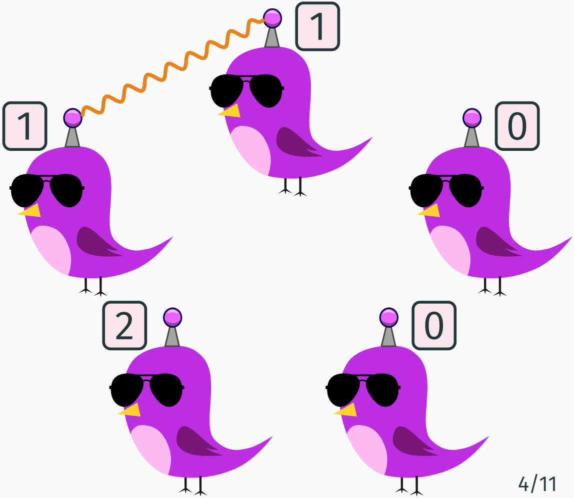

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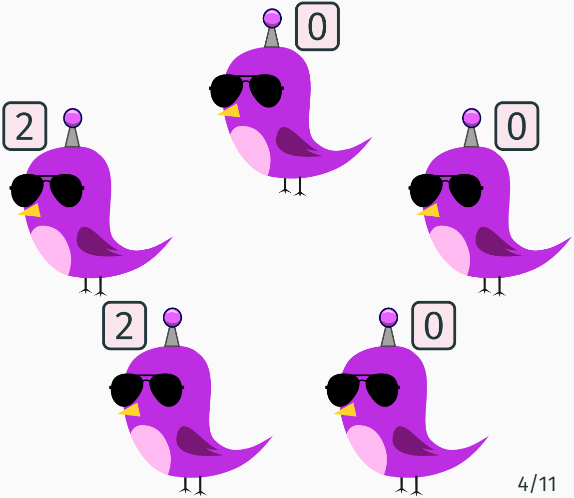

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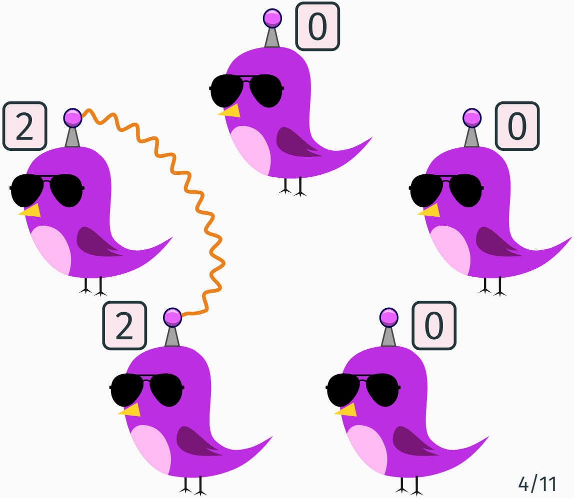

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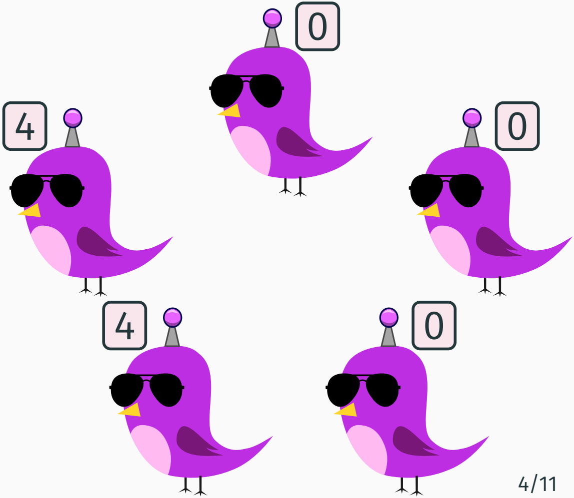

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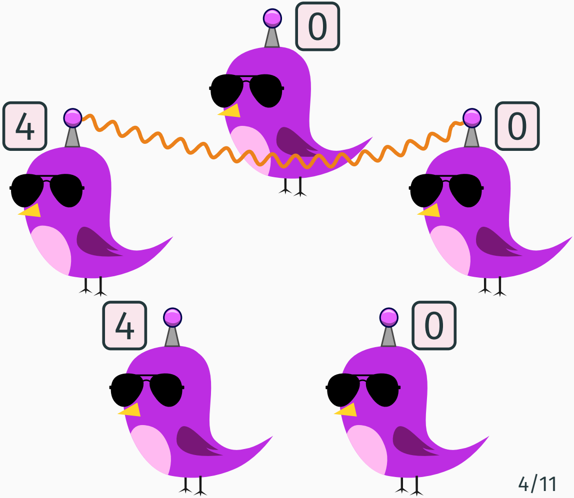

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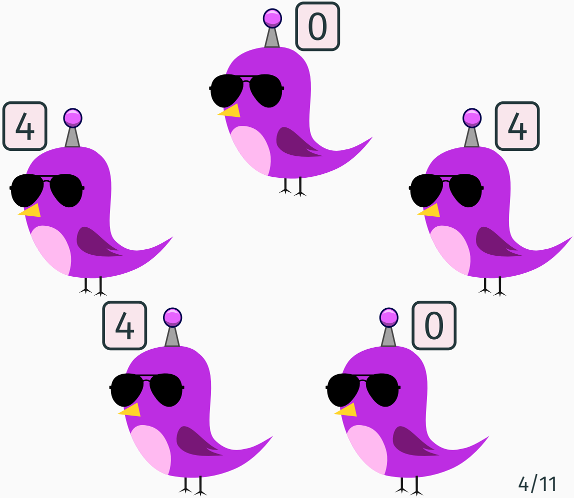

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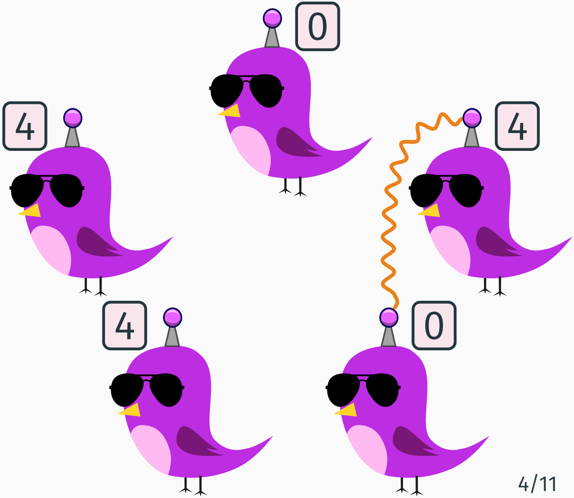

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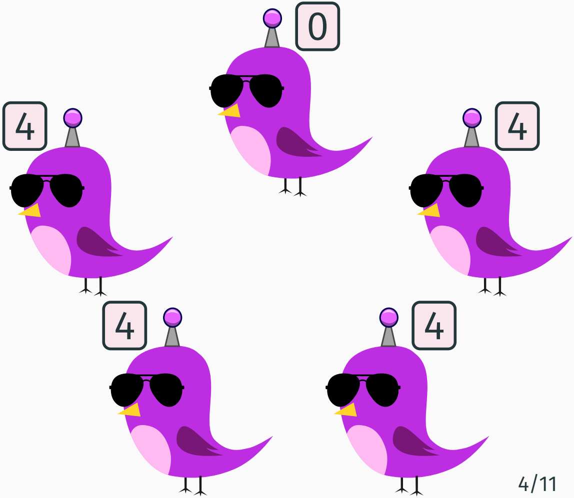

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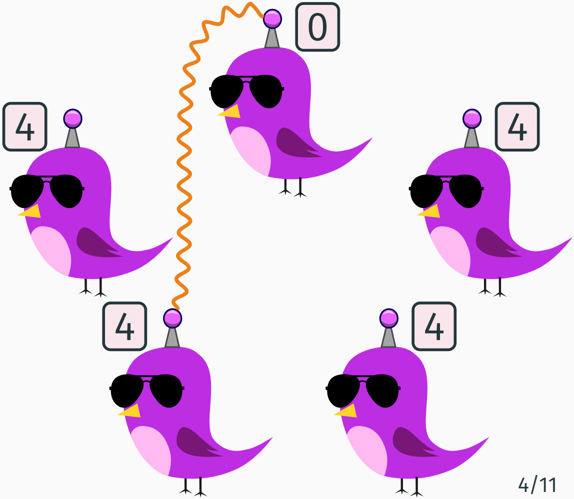

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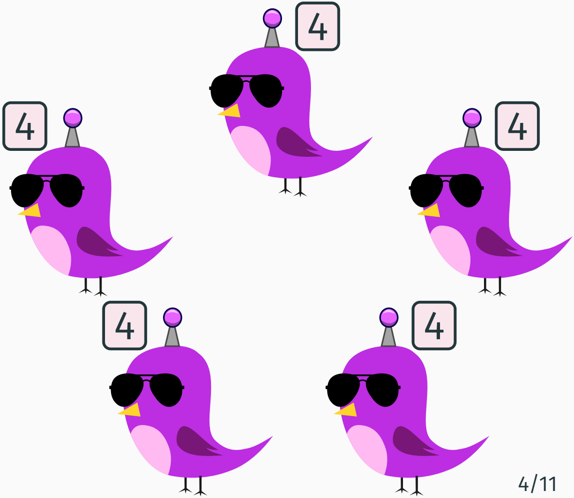

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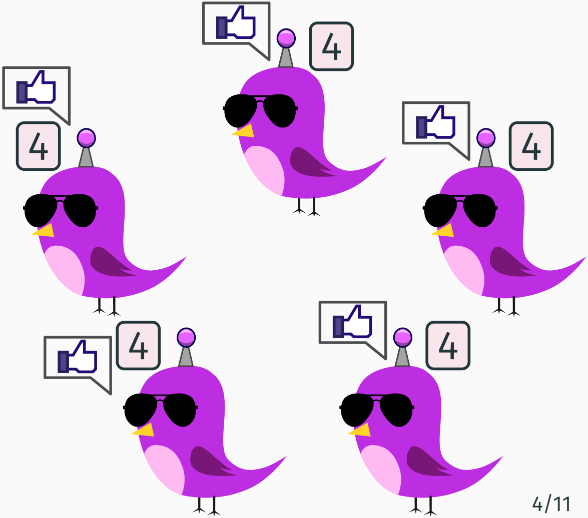
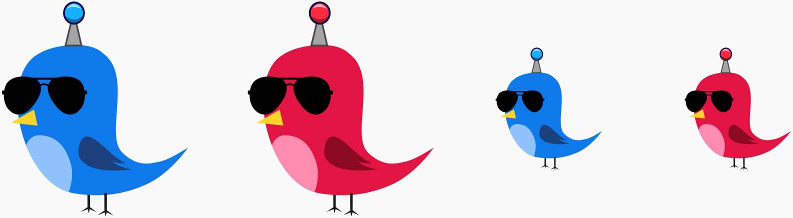


Demonstration

Population protocols: formal model

- *States*: finite set Q
- *Opinions*: $O : Q \to \{0, 1\}$
- *Initial states*: $I \subseteq Q$
- *Transitions*: $T \subseteq Q^2 \times Q^2$



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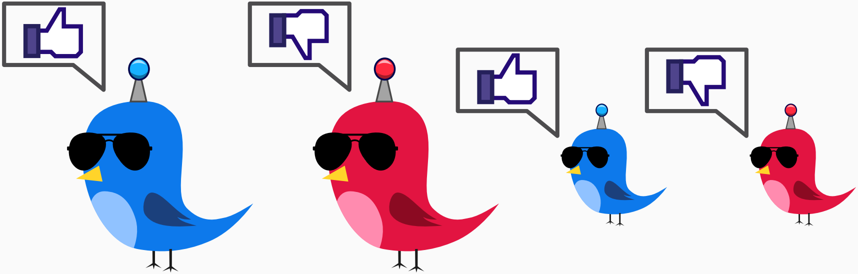

Population protocols: formal model

- *States*: finite set Q
- *Opinions*: $O : Q \to \{0, 1\}$
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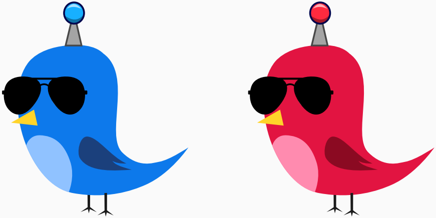

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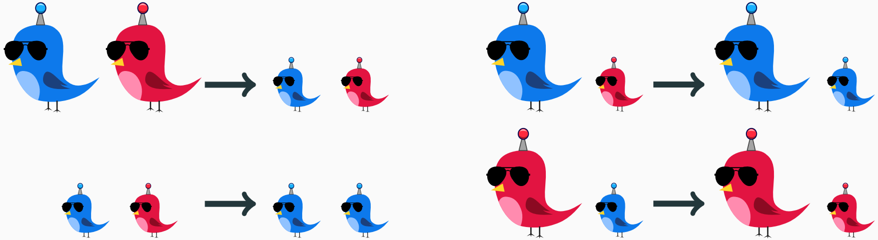
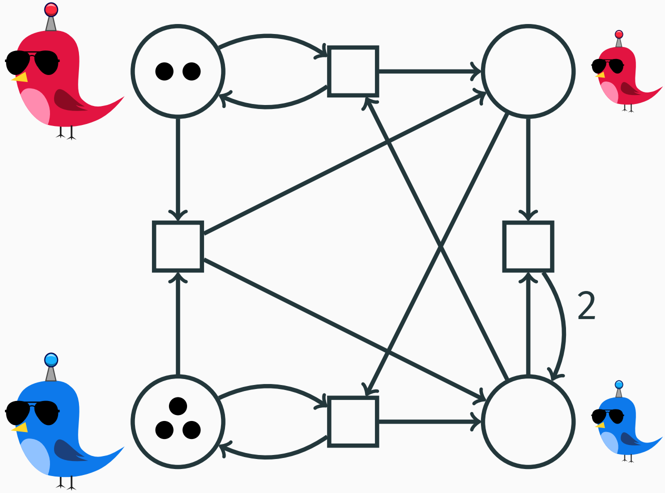
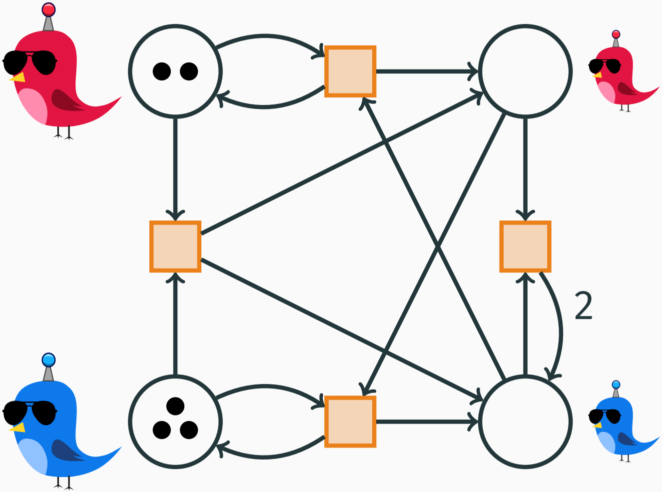


Protocols can be translated into Petri nets

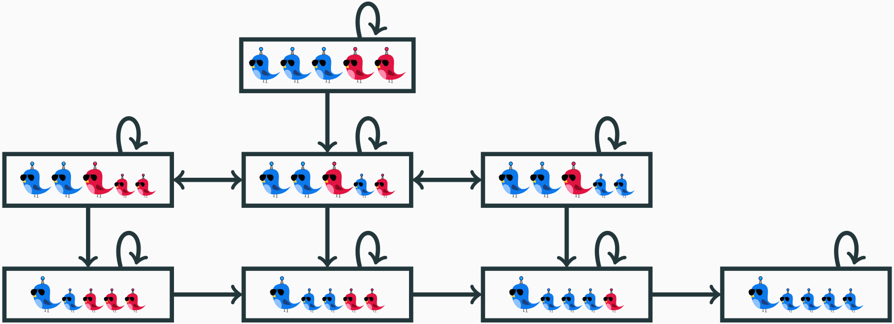


Protocols can be translated into Petri nets

conservative / bounded

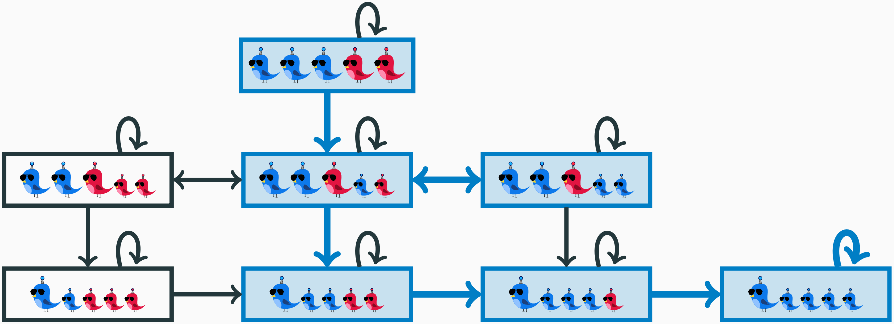


Reachability graph:



A *run* is an infinite path:

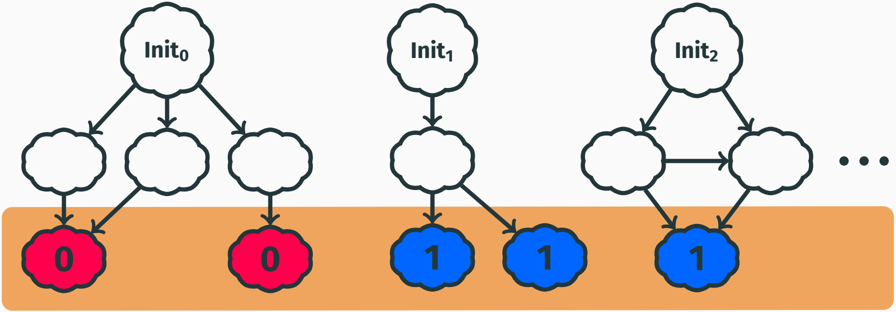
(pairs of agents are picked uniformly at random)



A protocol computes a predicate $\varphi\colon \mathbb{N}^I \to \{0, 1\}$

if runs reach common stable **consensus**

with probability 1



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Expressive power **Angluin, Aspnes, Eisenstat PODC'06**

Population protocols compute precisely predicates definable in Presburger arithmetic, *i.e.* FO($\mathbb{N}$, $+$, $<$)

Protocols speed

$$\mathbf{B}, \mathbf{R} \mapsto \mathbf{b}, \mathbf{r}$$

$$\mathbf{B}, \mathbf{r} \mapsto \mathbf{B}, \mathbf{b}$$

$$\mathbf{R}, \mathbf{b} \mapsto \mathbf{R}, \mathbf{r}$$

$$\mathbf{b}, \mathbf{r} \mapsto \mathbf{b}, \mathbf{b}$$

Computes correctly predicate #**B** ≥ #**R**

...but how fast?

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Computes correctly predicate $\#\mathbf{B} \geq \#\mathbf{R}$

...but how fast?

- **Natural to want protocols to be fast**
- **Upper bounds on speed useful since generally not possible to know whether a protocol has stabilized**

Protocols speed

$$\mathbf{B}, \mathbf{R} \mapsto \mathbf{b}, \mathbf{r}$$
$$\mathbf{B}, \mathbf{r} \mapsto \mathbf{B}, \mathbf{b}$$
$$\mathbf{R}, \mathbf{b} \mapsto \mathbf{R}, \mathbf{r}$$
$$\mathbf{b}, \mathbf{r} \mapsto \mathbf{b}, \mathbf{b}$$

Simulations show that it is slow when **R** has slight majority:

	Steps	Initial configuration
🟧	100000	{B: 7, R: 8}
🟩	7	{B: 3, R: 12}
🟩	27	{B: 4, R: 11}
🟧	100000	{B: 7, R: 8}
🟩	3	{B: 13, R: 2}

Protocols speed

$$\mathbf{B}, \mathbf{R} \mapsto \mathbf{T}, \mathbf{t} \qquad X, y \mapsto X, x \quad \text{for } x, y \in \{\mathbf{b}, \mathbf{r}, \mathbf{t}\}$$

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$$O(\mathbf{B}) = O(\mathbf{b}) = O(\mathbf{T}) = O(\mathbf{t}) = 1$$

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Alternative protocol

Protocols speed

$$\mathbf{B},\mathbf{R} \mapsto \mathbf{T},\mathbf{t} \qquad X,y \mapsto X,x \quad \text{for } x,y \in \{\mathbf{b},\mathbf{r},\mathbf{t}\}$$

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Is it faster?

Alternative protocol

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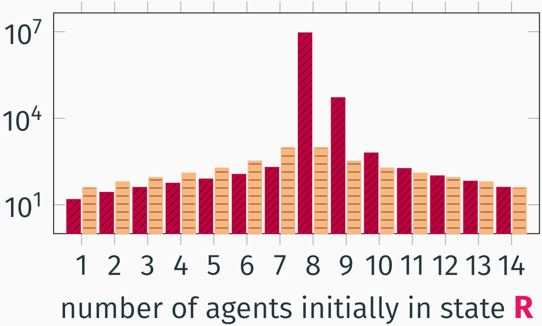
$X, y \mapsto X, x$ for $x, y \in \{\mathbf{b}, \mathbf{r}, \mathbf{t}\}$

Is it faster?

Yes, for size 15...

expected number of steps to stable consensus



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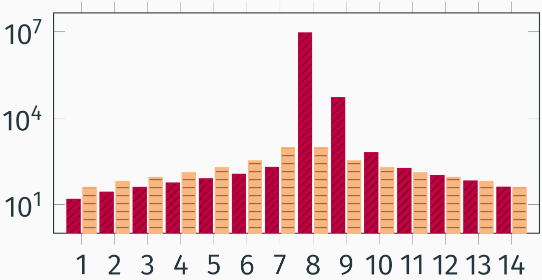
Obtained using PRISM

Clément et al. ICDCS'11, Offtermatt '17

expected number of steps to stable consensus



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B, **R**	$\mapsto$	**T**, **t**
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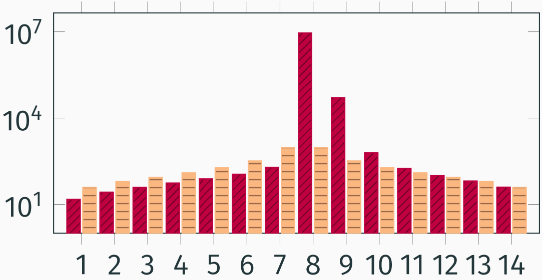
Our goal: analyze speed for all sizes

expected number of steps to stable consensus



number of agents initially in state **R**

Protocols speed: related work

- Any Presburger-definable predicate is computable in time $\mathcal{O}(n^2 \log n)$ Angluin *et al.* (PODC'04)
- Upper/lower bounds for majority and leader election
- Study of trade-offs between speed and number of states

  e.g.
  - Alistarh, Aspnes, Eisenstat, Gelashvili and Rivest (SODA'17)
  - Belleville, Doty and Soloveichik (ICALP'17)
  - Doty and Soloveichik (DISC'15), etc.

Definitions: probabilities

$$\mathbb{P}[\text{fire } p, q \mapsto p', q' \text{ in } C] = \begin{cases} \dfrac{2 \cdot C(p) \cdot C(q)}{n^2 - n} & \text{if } p \neq q \\[2ex] \dfrac{C(p) \cdot (C(p) - 1)}{n^2 - n} & \text{if } p = q \end{cases}$$

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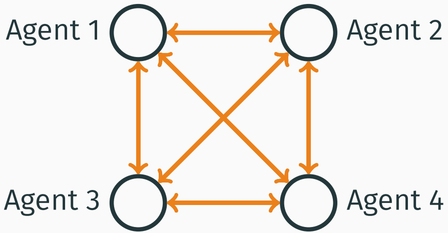

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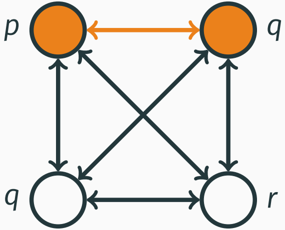

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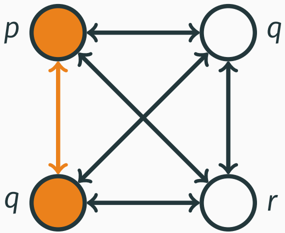

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$$\mathbb{P}[C \to C'] = \sum_{t \text{ s.t. } C \xrightarrow{t} C'} \mathbb{P}[\text{fire } t \text{ in } C]$$

$(Runs(C), \mathcal{F}, \mathbb{P}_C)$ is the probability space such that

- $\mathcal{F}$ is the σ-algebra generated by all

$$Runs(C_0, C_1, \ldots, C_k) = \{C = C_0 \to \cdots \to C_k \to \cdots\}$$

- $\mathbb{P}_C$ is the probability measure satisfying

$$\mathbb{P}_C(Runs(C_0, \ldots, C_k)) = \prod_{i=0}^{k-1} \mathbb{P}[C_i \to C_{i+1}]$$

Definitions: a simple temporal logic

$$C \models q \iff C(q) \geq 1$$

$$C \models q! \iff C(q) = 1$$

$$C \models Out_b \iff O(q) = b \text{ for every } q \models C$$

$$C \models \neg\varphi \iff C \not\models \varphi$$

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assigns to each run σ the smallest k s.t. $\sigma_k \models \varphi$, otherwise ∞

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Maximal expected termination time

We are interested in $time : \mathbb{N} \to \mathbb{N}$ where

$$time(n) = \max\{\mathbb{E}_C[Steps_{\Box Out_0 \vee \Box Out_1}] : C \text{ is initial and } |C| = n\}$$

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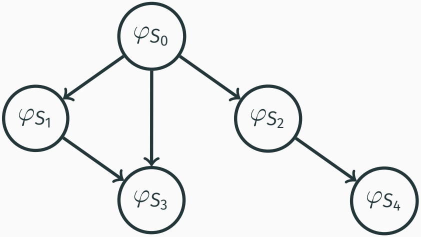
Our approach:

- Most protocols are naturally designed in stages
- Construct these stages automatically
- Derive bounds on expected running time
  from stages structure

Stage graphs

A *stage graph* is a directed acyclic graph $(\mathbb{S}, \to)$ such that

- every node $S \in \mathbb{S}$ is associated to a formula φ_S



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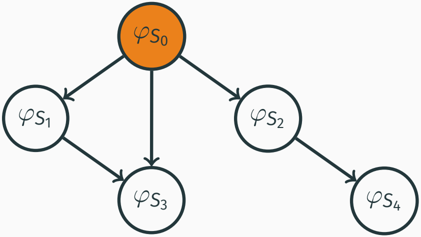

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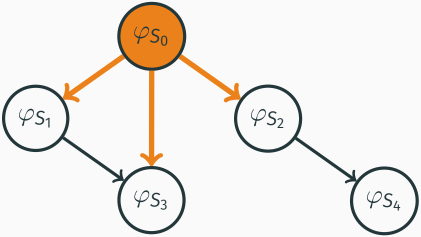

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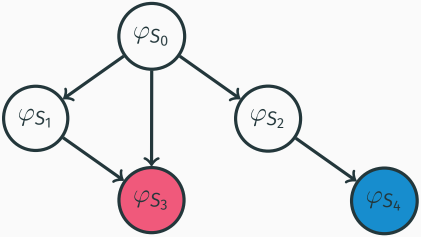
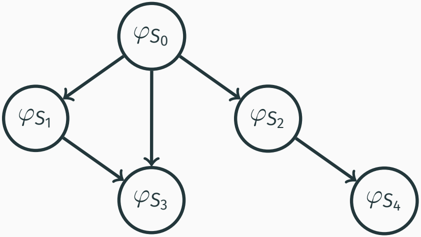
- $C \models \Diamond \bigvee_{S \rightarrow S'} \varphi_{S'}$ for every $S \in \mathbb{S}$ and $C \models \varphi_S$
- $C \models \varphi_S$ implies $C \models \Box Out_0 \vee \Box Out_1$ for every bottom $S \in \mathbb{S}$



Stage graphs

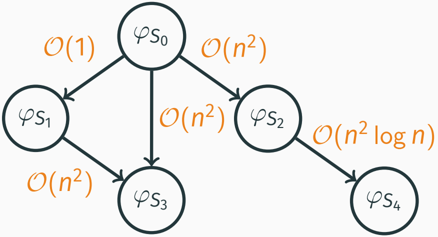
time(n) is bounded by the maximal expected number of steps to move from a stage to a successor



Stage graphs

time(n) is bounded by the maximal expected number of steps to move from a stage to a successor

For example, $time(n) \in O(n^2 \log n)$ if:



A procedure for computing stage graphs

$$\mathbf{B}, \mathbf{R} \mapsto \mathbf{T}, \mathbf{t}$$

$$\mathbf{B}, \mathbf{T} \mapsto \mathbf{B}, \mathbf{b}$$

$$\mathbf{R}, \mathbf{T} \mapsto \mathbf{R}, \mathbf{r}$$

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$$X, y \mapsto X, x$$

$$S_0 : (\mathbf{B} \vee \mathbf{R}) \wedge \bigwedge_{q \notin \{\mathbf{B}, \mathbf{R}\}} \neg q$$

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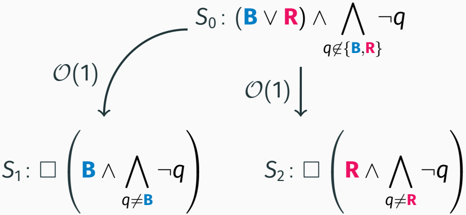

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X, y	$\mapsto$	X, x

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$\mathcal{O}(1)$ $\swarrow$ $\qquad$ $\mathcal{O}(1) \downarrow$

$$S_1 : \Box \left(\mathbf{B} \wedge \bigwedge_{q \neq \mathbf{B}} \neg q \right) \qquad S_2 : \Box \left(\mathbf{R} \wedge \bigwedge_{q \neq \mathbf{R}} \neg q \right)$$

Transformation graph

A procedure for computing stage graphs

- **B**, **R** $\mapsto$ **T**, **t**
- **B**, **T** $\mapsto$ **B**, **b**
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- **T**, **T** $\mapsto$ **T**, **t**
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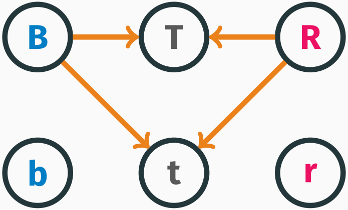

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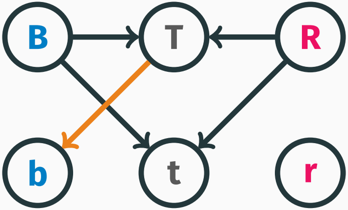

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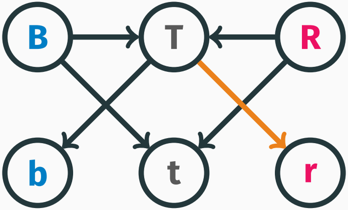

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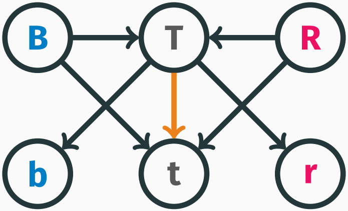
$\mathcal{O}(1)$ $\swarrow$ $\qquad$ $\mathcal{O}(1)$ $\downarrow$

$$S_1 : \Box \left(\mathbf{B} \wedge \bigwedge_{q \neq \mathbf{B}} \neg q \right) \qquad S_2 : \Box \left(\mathbf{R} \wedge \bigwedge_{q \neq \mathbf{R}} \neg q \right)$$



A procedure for computing stage graphs

$\mathbf{B}, \mathbf{R} \mapsto \mathbf{T}, \mathbf{t}$

$\mathbf{B}, \mathbf{T} \mapsto \mathbf{B}, \mathbf{b}$

$\mathbf{R}, \mathbf{T} \mapsto \mathbf{R}, \mathbf{r}$

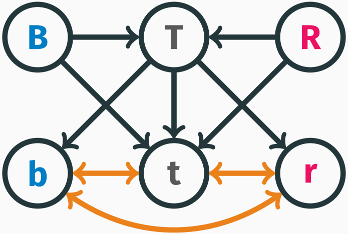
$\mathbf{T}, \mathbf{T} \mapsto \mathbf{T}, \mathbf{t}$

$X, y \mapsto X, x$

$S_0 : (\mathbf{B} \vee \mathbf{R}) \wedge \bigwedge_{q \notin \{\mathbf{B},\mathbf{R}\}} \neg q$

$\mathcal{O}(1)$ $\qquad$ $\mathcal{O}(1)$

$S_1 : \square \left(\mathbf{B} \wedge \bigwedge_{q \neq \mathbf{B}} \neg q \right)$ $\qquad$ $S_2 : \square \left(\mathbf{R} \wedge \bigwedge_{q \neq \mathbf{R}} \neg q \right)$



A procedure for computing stage graphs

B, **R**	$\mapsto$	**T**, **t**
B, **T**	$\mapsto$	**B**, **b**
R, **T**	$\mapsto$	**R**, **r**
T, **T**	$\mapsto$	**T**, **t**
X, y	$\mapsto$	X, x

$S_0 : (\mathbf{B} \vee \mathbf{R}) \wedge \bigwedge_{q \notin \{\mathbf{B},\mathbf{R}\}} \neg q$

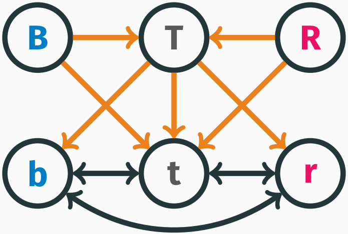
$\mathcal{O}(1)$

$\mathcal{O}(1)$

$S_1 : \Box \left(\mathbf{B} \wedge \bigwedge_{q \neq \mathbf{B}} \neg q \right)$

$S_2 : \Box \left(\mathbf{R} \wedge \bigwedge_{q \neq \mathbf{R}} \neg q \right)$

Will become permanently disabled almost surely



A procedure for computing stage graphs

B, **R**	$\mapsto$	**T**, **t**
B, **T**	$\mapsto$	**B**, **b**
R, **T**	$\mapsto$	**R**, **r**
T, **T**	$\mapsto$	**T**, **t**

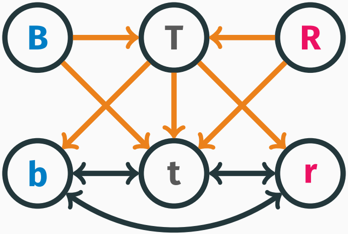
$X, y \mapsto X, x$

$$S_0 : (\mathbf{B} \vee \mathbf{R}) \wedge \bigwedge_{q \notin \{\mathbf{B},\mathbf{R}\}} \neg q$$

$\mathcal{O}(1)$ $\mathcal{O}(1)$

$$S_1 : \Box \left(\mathbf{B} \wedge \bigwedge_{q \neq \mathbf{B}} \neg q \right) \qquad S_2 : \Box \left(\mathbf{R} \wedge \bigwedge_{q \neq \mathbf{R}} \neg q \right)$$

$$S_3 : \Box\, [(\neg\mathbf{B} \vee \neg\mathbf{R}) \wedge (\neg\mathbf{B} \vee \neg\mathbf{T}) \wedge (\neg\mathbf{R} \vee \neg\mathbf{T}) \wedge (\neg\mathbf{T} \vee \mathbf{T}!)] \wedge ((\mathbf{B} \wedge \mathbf{b}) \vee (\mathbf{R} \wedge \mathbf{r}) \vee (\mathbf{T} \wedge \mathbf{t}))$$



A procedure for computing stage graphs

$\mathbf{B}, \mathbf{R} \mapsto \mathbf{T}, \mathbf{t}$

$\mathbf{B}, \mathbf{T} \mapsto \mathbf{B}, \mathbf{b}$

$\mathbf{R}, \mathbf{T} \mapsto \mathbf{R}, \mathbf{r}$

$\mathbf{T}, \mathbf{T} \mapsto \mathbf{T}, \mathbf{t}$

$X, y \mapsto X, x$

$$S_0 : (\mathbf{B} \vee \mathbf{R}) \wedge \bigwedge_{q \notin \{\mathbf{B},\mathbf{R}\}} \neg q$$

$\mathcal{O}(1)$

$\mathcal{O}(1)$

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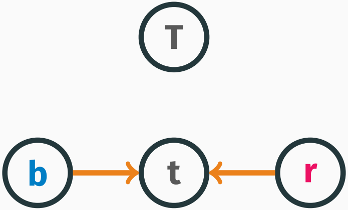
$$S_3 : \Box \left[(\neg \mathbf{B} \vee \neg \mathbf{R}) \wedge (\neg \mathbf{B} \vee \neg \mathbf{T}) \wedge (\neg \mathbf{R} \vee \neg \mathbf{T}) \wedge (\neg \mathbf{T} \vee \mathbf{T}!) \right] \wedge ((\mathbf{B} \wedge \mathbf{b}) \vee (\mathbf{R} \wedge \mathbf{r}) \vee (\mathbf{T} \wedge \mathbf{t}))$$



A procedure for computing stage graphs

$\mathbf{B}, \mathbf{R} \mapsto \mathbf{T}, \mathbf{t}$

$\mathbf{B}, \mathbf{T} \mapsto \mathbf{B}, \mathbf{b}$

$\mathbf{R}, \mathbf{T} \mapsto \mathbf{R}, \mathbf{r}$

$\mathbf{T}, \mathbf{T} \mapsto \mathbf{T}, \mathbf{t}$

$X, y \mapsto X, x$

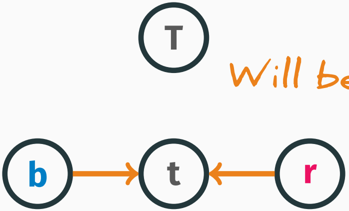
$$S_0 : (\mathbf{B} \vee \mathbf{R}) \wedge \bigwedge_{q \notin \{\mathbf{B},\mathbf{R}\}} \neg q$$

$\mathcal{O}(1)$

$\mathcal{O}(1)$

$$S_1 : \Box \left(\mathbf{B} \wedge \bigwedge_{q \neq \mathbf{B}} \neg q \right) \qquad S_2 : \Box \left(\mathbf{R} \wedge \bigwedge_{q \neq \mathbf{R}} \neg q \right)$$

$$S_3 : \Box \left[(\neg \mathbf{B} \vee \neg \mathbf{R}) \wedge (\neg \mathbf{B} \vee \neg \mathbf{T}) \wedge (\neg \mathbf{R} \vee \neg \mathbf{T}) \wedge (\neg \mathbf{T} \vee \mathbf{T}!) \right] \wedge$$
$$((\mathbf{B} \wedge \mathbf{b}) \vee (\mathbf{R} \wedge \mathbf{r}) \vee (\mathbf{T} \wedge \mathbf{t}))$$



Will become permanently disabled almost surely

A procedure for computing stage graphs

$\mathbf{B}, \mathbf{R} \mapsto \mathbf{T}, \mathbf{t}$

$\mathbf{B}, \mathbf{T} \mapsto \mathbf{B}, \mathbf{b}$

$\mathbf{R}, \mathbf{T} \mapsto \mathbf{R}, \mathbf{r}$

$\mathbf{T}, \mathbf{T} \mapsto \mathbf{T}, \mathbf{t}$

$X, y \mapsto X, x$

$$S_0 : (\mathbf{B} \vee \mathbf{R}) \wedge \bigwedge_{q \notin \{\mathbf{B},\mathbf{R}\}} \neg q$$

$\mathcal{O}(1)$

$\mathcal{O}(1)$

$$S_1 : \Box \left(\mathbf{B} \wedge \bigwedge_{q \neq \mathbf{B}} \neg q \right) \qquad S_2 : \Box \left(\mathbf{R} \wedge \bigwedge_{q \neq \mathbf{R}} \neg q \right)$$

$$S_3 : \Box \left[(\neg \mathbf{B} \vee \neg \mathbf{R}) \wedge (\neg \mathbf{B} \vee \neg \mathbf{T}) \wedge (\neg \mathbf{R} \vee \neg \mathbf{T}) \wedge (\neg \mathbf{T} \vee \mathbf{T!}) \right] \wedge ((\mathbf{B} \wedge \mathbf{b}) \vee (\mathbf{R} \wedge \mathbf{r}) \vee (\mathbf{T} \wedge \mathbf{t}))$$

T

t

$$S_6 : \Box \left(\mathbf{T!} \wedge \mathbf{t} \wedge \bigwedge_{q \notin \{\mathbf{T},\mathbf{t}\}} \neg q \right)$$

A procedure for computing stage graphs

$\mathbf{B}, \mathbf{R} \mapsto \mathbf{T}, \mathbf{t}$

$\mathbf{B}, \mathbf{T} \mapsto \mathbf{B}, \mathbf{b}$

$\mathbf{R}, \mathbf{T} \mapsto \mathbf{R}, \mathbf{r}$

$\mathbf{T}, \mathbf{T} \mapsto \mathbf{T}, \mathbf{t}$

$X, y \mapsto X, x$

$$S_0 : (\mathbf{B} \vee \mathbf{R}) \wedge \bigwedge_{q \notin \{\mathbf{B},\mathbf{R}\}} \neg q$$

$\mathcal{O}(1)$ $\mathcal{O}(1)$

$$S_1 : \Box \left(\mathbf{B} \wedge \bigwedge_{q \neq \mathbf{B}} \neg q \right) \qquad S_2 : \Box \left(\mathbf{R} \wedge \bigwedge_{q \neq \mathbf{R}} \neg q \right)$$

$$S_3 : \Box \left[(\neg \mathbf{B} \vee \neg \mathbf{R}) \wedge (\neg \mathbf{B} \vee \neg \mathbf{T}) \wedge (\neg \mathbf{R} \vee \neg \mathbf{T}) \wedge (\neg \mathbf{T} \vee \mathbf{T}!) \right] \wedge ((\mathbf{B} \wedge \mathbf{b}) \vee (\mathbf{R} \wedge \mathbf{r}) \vee (\mathbf{T} \wedge \mathbf{t}))$$

$$S_6 : \Box \left(\mathbf{T}! \wedge \mathbf{t} \wedge \bigwedge_{q \notin \{\mathbf{T},\mathbf{t}\}} \neg q \right)$$

$$\begin{aligned} \mathbb{E}_C[Steps_{\neg \mathbf{b} \wedge \neg \mathbf{r}}] &\leq \sum_{i=1}^{C(\mathbf{b})+C(\mathbf{r})} \frac{n^2}{2 \cdot C(\mathbf{T}) \cdot i} \\ &\leq \sum_{i=1}^{n} \frac{n^2}{i} \\ &\leq \alpha \cdot n^2 \cdot \log n \end{aligned}$$

A procedure for computing stage graphs

$$\mathbf{B}, \mathbf{R} \mapsto \mathbf{T}, \mathbf{t}$$
$$\mathbf{B}, \mathbf{T} \mapsto \mathbf{B}, \mathbf{b}$$
$$\mathbf{R}, \mathbf{T} \mapsto \mathbf{R}, \mathbf{r}$$
$$\mathbf{T}, \mathbf{T} \mapsto \mathbf{T}, \mathbf{t}$$
$$X, y \mapsto X, x$$

$$S_0 : (\mathbf{B} \vee \mathbf{R}) \wedge \bigwedge_{q \notin \{\mathbf{B},\mathbf{R}\}} \neg q$$

$\mathcal{O}(1)$

$\mathcal{O}(1)$

$$S_1 : \Box \left(\mathbf{B} \wedge \bigwedge_{q \neq \mathbf{B}} \neg q \right) \qquad S_2 : \Box \left(\mathbf{R} \wedge \bigwedge_{q \neq \mathbf{R}} \neg q \right)$$

$$S_3 : \Box \left[(\neg \mathbf{B} \vee \neg \mathbf{R}) \wedge (\neg \mathbf{B} \vee \neg \mathbf{T}) \wedge (\neg \mathbf{R} \vee \neg \mathbf{T}) \wedge (\neg \mathbf{T} \vee \mathbf{T}!) \right] \wedge ((\mathbf{B} \wedge \mathbf{b}) \vee (\mathbf{R} \wedge \mathbf{r}) \vee (\mathbf{T} \wedge \mathbf{t}))$$

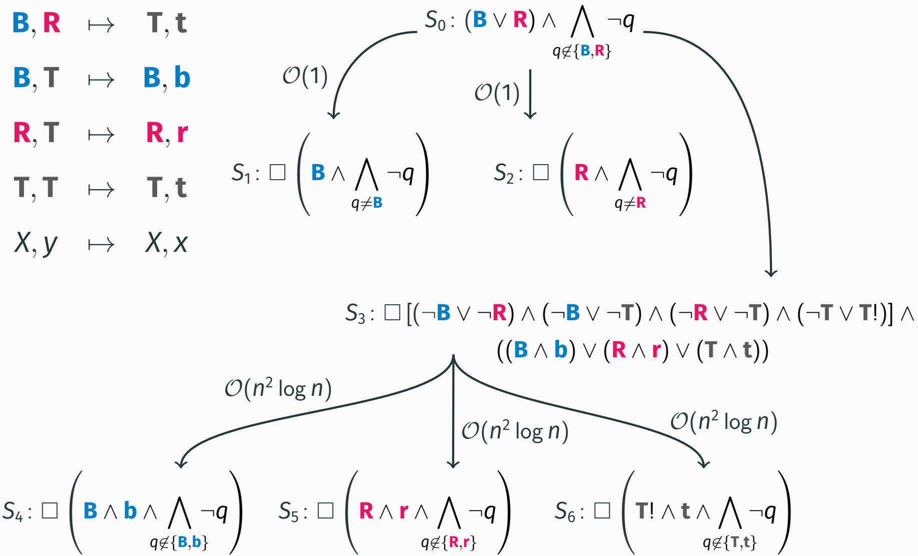
$\mathcal{O}(n^2 \log n)$

$$S_6 : \Box \left(\mathbf{T}! \wedge \mathbf{t} \wedge \bigwedge_{q \notin \{\mathbf{T},\mathbf{t}\}} \neg q \right)$$

$$\mathbb{E}_C[Steps_{\neg \mathbf{b} \wedge \neg \mathbf{r}}] \leq \sum_{i=1}^{C(\mathbf{b})+C(\mathbf{r})} \frac{n^2}{2 \cdot C(\mathbf{T}) \cdot i} \leq \sum_{i=1}^{n} \frac{n^2}{i} \leq \alpha \cdot n^2 \cdot \log n$$

A procedure for computing stage graphs

$$\mathbf{B}, \mathbf{R} \mapsto \mathbf{T}, \mathbf{t}$$
$$\mathbf{B}, \mathbf{T} \mapsto \mathbf{B}, \mathbf{b}$$
$$\mathbf{R}, \mathbf{T} \mapsto \mathbf{R}, \mathbf{r}$$
$$\mathbf{T}, \mathbf{T} \mapsto \mathbf{T}, \mathbf{t}$$
$$X, y \mapsto X, x$$



A procedure for computing stage graphs

B, **R**	$\mapsto$	**T**, **t**
B, **T**	$\mapsto$	**B**, **b**
R, **T**	$\mapsto$	**R**, **r**
T, **T**	$\mapsto$	**T**, **t**
X, y	$\mapsto$	X, x

$$S_0 : (\mathbf{B} \vee \mathbf{R}) \wedge \bigwedge_{q \notin \{\mathbf{B},\mathbf{R}\}} \neg q$$

$\mathcal{O}(1)$

$$S_1 : \Box \left(\mathbf{B} \wedge \bigwedge_{q \neq \mathbf{B}} \neg q \right)$$

$\mathcal{O}(1)$

$$S_2 : \Box \left(\mathbf{R} \wedge \bigwedge_{q \neq \mathbf{R}} \neg q \right)$$

$\mathcal{O}(n^2)$

$$S_3 : \Box \left[(\neg\mathbf{B} \vee \neg\mathbf{R}) \wedge (\neg\mathbf{B} \vee \neg\mathbf{T}) \wedge (\neg\mathbf{R} \vee \neg\mathbf{T}) \wedge (\neg\mathbf{T} \vee \mathbf{T}!) \right] \wedge ((\mathbf{B} \wedge \mathbf{b}) \vee (\mathbf{R} \wedge \mathbf{r}) \vee (\mathbf{T} \wedge \mathbf{t}))$$

$\mathcal{O}(n^2 \log n)$

$$S_4 : \Box \left(\mathbf{B} \wedge \mathbf{b} \wedge \bigwedge_{q \notin \{\mathbf{B},\mathbf{b}\}} \neg q \right)$$

$\mathcal{O}(n^2 \log n)$

$$S_5 : \Box \left(\mathbf{R} \wedge \mathbf{r} \wedge \bigwedge_{q \notin \{\mathbf{R},\mathbf{r}\}} \neg q \right)$$

$\mathcal{O}(n^2 \log n)$

$$S_6 : \Box \left(\mathbf{T}! \wedge \mathbf{t} \wedge \bigwedge_{q \notin \{\mathbf{T},\mathbf{t}\}} \neg q \right)$$

A procedure for computing stage graphs

Φ: propositional formula describing current configurations

π: set of permanently present/absent states

$\mathcal{T}$: set of permanently disabled transitions

Successors computed by enriching
π through trap/siphon-like analysis and
$\mathcal{T}$ and Φ from transformation graph

A procedure for computing stage graphs

Φ: propositional formula describing current configurations

π: set of permanently present/absent states

$\mathcal{T}$: set of permanently disabled transitions

Successors computed by enriching
π through trap/siphon-like analysis and
$\mathcal{T}$ and Φ from transformation graph

Experimental results

- Prototype implemented in python™ + Microsoft Z3
- Can report: $\mathcal{O}(1), \mathcal{O}(n^2), \mathcal{O}(n^2 \log n), \mathcal{O}(n^3), \mathcal{O}(\text{poly}(n))$ or $\mathcal{O}(\exp(n))$
- Tested on various protocols from the literature

Experimental results

Protocol: φ / params.	$\|Q\|$	$\|T\|$	Stages	Bound	Time
$x_1 \vee \ldots \vee x_n$ [b]	2	1	5	$n^2 \log n$	0.1
$x \geq y$ [a]	6	10	23	$n^2 \log n$	0.9
$x \geq y$ [c]	4	3	9	$n^2 \log n$	0.2
$x \geq y$ [c]	4	4	11	$\exp(n)$	0.3
Threshold [a]: $x \geq c$					
$c = 5$	6	21	26	n^3	0.8
$c = 15$	16	136	66	n^3	12.1
$c = 25$	26	351	106	n^3	58.0
$c = 35$	36	666	146	n^3	222.3
$c = 45$	46	1081	186	n^3	495.3
$c = 55$	56	1596	—	—	T/O
Logarithmic threshold: $x \geq c$					
$c = 7$	6	14	34	n^3	1.9
$c = 31$	10	34	130	n^3	6.1
$c = 127$	14	62	514	n^3	39.4
$c = 1023$	20	119	4098	n^3	395.7
$c = 4095$	24	167	—	—	T/O

Protocol: φ / params.	$\|Q\|$	$\|T\|$	Stages	Bound	Time
Threshold [b]: $x \geq c$					
$c = 5$	6	9	54	n^3	2.5
$c = 7$	8	13	198	n^3	11.3
$c = 10$	11	19	1542	n^3	83.9
$c = 13$	14	25	12294	n^3	816.4
$c = 15$	16	29	—	—	T/O
Average-and-conquer [d]: $x \geq y$ (param. m, d)					
$m = 3, d = 1$	6	21	41	$n^2 \log n$	2.0
$m = 3, d = 2$	8	36	1948	$n^2 \log n$	98.7
$m = 5, d = 1$	8	36	1870	n^3	80.1
$m = 5, d = 2$	10	55	—	—	T/O
Remainder [a]: $\sum_{1 \leq i < m} i \cdot x_i \equiv 0 \pmod c$					
$c = 5$	7	25	225	$n^2 \log n$	12.5
$c = 7$	9	42	1351	$n^2 \log n$	88.9
$c = 9$	11	63	7035	$n^2 \log n$	544.0
$c = 10$	12	75	—	—	T/O
Linear inequalities [a]					
$-x_1 + x_2 < 0$	12	57	21	n^3	3.0
$-x_1 + x_2 < 1$	20	155	131	n^3	30.3
$-x_1 + x_2 < 2$	28	301	—	—	T/O

[a] Angluin *et al.* 2006

[b] Clément *et al.* 2011

[c] Draief *et al.* 2012

[d] Alistarh *et al.* 2015

Conclusion: summary

- First procedure providing *asymptotic* upper bounds on expected termination time
- Approach promising in practice
- New crucial notions: stage graphs and transformation graphs

Conclusion: future work

- Is our procedure "weakly complete"? *i.e.* for every φ, is there a protocol for φ analyzable by our procedure?
- Approach can be used for verification?
- How to compute lower bounds?

Thank you!

Merci!